Logic

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1 Propositional Logic

Philosophers still debate about what a proposition really means. As a complete beginner, I mention some interpretations of it, but I by no means claim that this is the definitive definition.

Definition 1.1 (Possible World)

A **possible world** is a complete and consistent way the world is or could have been.

The **language** of propositional logic consists of just two things: propositions and connectives.

Definition 1.2 (Proposition)

A **proposition** does not have a formal definition, but we can describe it in the following ways.

1. They can be understood as an indicator function $f : W \to \{T, F\}^a$ that takes in a possible world and returns a truth value. We can also model it with the preimage of f under T, i.e. the characteristic set of f.
2. They deal with **statements**, which are defined as declarative sentences having a truth value.

Propositions are either true or false.

Example 1.1 ()

The proposition that *the sky is blue* is represented as the function that returns T for every possible world where the sky is blue.

These declarative sentences are contrasted with questions, such as *how are you doing?* and imperative statements such as *please run my models.* Such non-declarative sentences have no truth value.

A statement can contain one or more other statements as parts. For example, compound sentences form simpler sentences.

Definition 1.3 (Connectives)

Statements are combined with **logical connectives**.

Connective	**Symbols**
AND	$A \wedge B$, $A \cdot B$, AB, $A\&B$, $A\&\&B$
OR	$A \vee B$, $A + B$, $A \mid B$, $A \parallel B$
NOT	$\neg A$, $-A$, $\overline{A}$, $\sim A$
NAND	$\overline{A \wedge B}$, $A \mid B$, $\overline{A \cdot B}$
NOR	$\overline{A \vee B}$, $A \downarrow B$, $\overline{A + B}$
XOR	$A \veebar B$, $A \oplus B$
XNOR	$A \odot B$
IMPLIES	$A \Rightarrow B$, $A \supset B$, $A \rightarrow B$
EQUIVALENT	$A \equiv B$, $A \Leftrightarrow B$, $A \leftrightarrow B$
NONEQUIVALENT	$A \not\equiv B$, $A \not\Leftrightarrow B$, $A \not\leftrightarrow B$

Table 1: Logical Connectives and Their Symbols

[a] T, F stands for True, False.

Definition 1.4 (Propositional Formula)

Propositions, represented by letters and denoted **propositional variables**, along with these symbols for connectives, combine to make a **propositional formula**.

Propositional logic is not concerned with the structures of propositions beyond the point where they cannot be decomposed any more by logical connectives.

1.1 Arguments

At this point we may look at a set of propositions $P_1, \ldots, P_n$ and try come to a logical conclusion Q. This is called an argument.

Definition 1.5 (Argument)

Let P be a set of propositions, called the **premises**. Let Q be a proposition, called the **conclusion**. Then an **argument** is an attempt to deduce Q from P. It is written in the forms

1. If P, then Q.
2. $P \implies Q$

An argument is **valid** if and only if

1. It is necessary that if P is true, Q is true.
2. It is impossible for P to be true, while Q is false.

Example 1.2 ()

The following is an argument.

1. If *it is raining*, then *it is cloudy*.
2. *It is raining*.
3. *Therefore it is cloudy*.

Logic in general aims to specify valid arguments. This is done by defining a valid argument as one in which its conclusion is a logical consequence of its premises. Determining whether a proposition is a a logical consequence of another proposition is the process of **deductive argument**, which has rules. These rules, called **rules of inference**, determines the "legal moves" from one or more premises to the conclusion. We give 2 familiar ones.

Definition 1.6 (Modus Ponens)

Modus ponens is a deductive argument form and rule of inference.[a] The argument states that given the premises

1. $P \implies Q$
2. P

Then our conclusion is Q.

The next one is the familiar statement that a statement is equivalent to its contrapositive.

Definition 1.7 (Modus Tollens)

Modus tollens is a deductive argument form and a rule of inference. The argument states that given the premises

1. $P \implies Q$
2. $\neg Q$

[a] In some literature it is treated as an axiom, though most people think of it as a rule.

Then our conclusion is not P.

2 First-Order Logic

In propositional logic, we deal with simple declarative propositions. **First-order logic** extends this by covering predicates and quantification. Let's motivate them.

We can think of predicates as properties. If we say *Socrates is a philosopher* and *Plato is a philosopher*, in propositional logic both these statements, represented as P and Q, as utterances that are either true or false, and they are completely independent from one another. However, we may want to view them as an application of a predicate $*$ *is a philosopher* on the entities *Socrates* and *Plato*. This motives the formalism of the domain of discourse and the predicate.

Definition 2.1 (Domain of Discourse)

Given an individual x, its **domain of discourse** is the set over which certain variables of interest in some formal treatment may range.

Definition 2.2 (Predicate)

A **predicate** P is a symbol that represents a property or a relation of a certain individual x in a domain of discourse. Using predicates, $P(x)$ can be viewed as a proposition about the individual x.

Note that a predicate itself is not a proposition, since saying $*$ *is a philosopher* doesn't have any truth or false meaning to it, akin to a sentence fragment. But it is a placeholder $P(\cdot)$ upon which if an individual x is put in, it makes sense to ask whether $P(x)$ is true.

Definition 2.3 (Formula)

A **formula** is a string of propositions, connectives, predicates, and variables ϕ that turns into a proposition once all free variables have been instantiated.

With predicates alone, all we have really done is add notational convenience. However, if we want to state a proposition not just about x, but its domain of discourse, then we can use quantifiers.

Definition 2.4 (Quantifier)

A **quantifier** is an operator that specifies how many individuals in the domain of source satisfy a proposition. The two most used quantifiers are

1. *Universal Quantification.* $\forall$, which means *for every.*
2. *Existential Quantification.* $\exists$, which means *there exists.*

These quantifiers are additional symbols in our language $\mathcal{L}$. If we add the equality symbol, we get first-order logic with equality.

Axiom 2.1 (Equality)

Equality is a primitive logical symbol which is always interpreted as the real equality relation between members of the domain of discourse. These equality axioms are:

1. *Reflexivity.* For each variable x, $x = x$.
2. *Substitution for Functions.* For all variables x and y, and any function symbol f,

$$x = y \implies f(x) = f(y) \tag{1}$$

3. *Substitution for Formulas.* For any variables x and y, and any formula $\phi(z)$ with free variable z, then
$$x = y \implies (\phi(x) \implies \phi(y)) \tag{2}$$

Symmetry and transitivity follow from the axioms above.

Ordinary first-order interpretations have a single domain of discourse over which all quantifiers range. **Many-sorted first-order logic**, or **typed first-order logic** allows variables to have different **sorts** or **types**, i.e. coming from different domains.

2.1 Exercises

Exercise 2.1 (Shifrin Abstract Algebra Appendix 1.1)

Negate the following sentences; in each case, indicate whether the original sentence or its negation is a true statement. Be sure to move the “not" through all the quantifiers.

1. For every integer $n \geq 2$, the number $2^n - 1$ is prime.
2. There exists a real number M so that for all real numbers t, $|\sin t| \leq M$.
3. For every real number $x > 0$, there exists a real number $y > 0$ so that $xy > 1$.

Solution 2.1

Listed.

1. *Negation.* For at least one $n \geq 2$, the number $2^n - 1$ is composite (not prime). The negation is true. Consider $n = 4 \implies 2^4 - 1 = 15 = 3 \cdot 5$.
2. *Negation.* There exists no real number M such that for all real numbers t, $|\sin t| \leq M$. The original is true. Pick $M = 1$, and by definition $|\sin t| \leq 1$.
3. *Negation.* For at least one real number $x > 0$, there exists no real number $y > 0$ so that $xy > 1$. The original is true. Given a real number $x > 0$, choose $y = \frac{1}{x} + 1$. Then,
$$xy = x\left(\frac{1}{x} + 1\right) = 1 + x > 1 \tag{3}$$
where the steps follow from the ordered field properties of $\mathbb{R}$.

Exercise 2.2 (Shifrin Abstract Algebra Appendix 1.4)

Suppose n is an odd integer. Prove:

1. The equation $x^2 + x - n = 0$ has no solution $x \in \mathbb{Z}$.
2. Prove that for any $m \in \mathbb{Z}$, the equation $x^2 + 2mx + 2n = 0$ has no solution $x \in \mathbb{Z}$.

Solution 2.2

We prove by contradiction. Assume such a solution x exists for odd n. We consider the two cases where x

1. is even.
$$\begin{aligned} x \text{ is even} &\implies x \equiv 0 \pmod 2 && (4) \\ &\implies x^2 + x \equiv 0 \pmod 2 && (5) \\ &\implies x^2 + x - n \equiv 1 \pmod 2 && (6) \end{aligned}$$

2. is odd.

$$\begin{aligned} x \text{ is odd} &\implies x \equiv 1 \pmod 2 && (7) \\ &\implies x^2 + x \equiv 1 + 1 \equiv 0 (\text{mod} 2) && (8) \\ &\implies x^2 + x - n \equiv 1 (\text{mod} 2) && (9) \end{aligned}$$

Both cases result in the quadratic expression lying in the equivalence class [1] and thus cannot be 0. This contradicts our assumption that it is a solution. We prove by contradiction. Assume a solution x exists for odd n. Note that since $x^2 + 2mx + 2n \equiv x^2 \equiv 0 \pmod 2$, this implies that $x \equiv 0 \pmod 2$.[a] Therefore, we can write $x = 2x'$ for some $x' \in \mathbb{Z}$, our assumption is equivalent to the existence of x'. Substituting this gives

$$4x'^2 + 4mx' + 2n = 0 \iff 2x'^2 + 2mx' + n = 0 \tag{10}$$

Since $2x'^2 + 2mx'$ is even, n must be even as well, which contradicts our assumption that n is odd.

3 Second-Order Logic

First order logic can quantify over individuals, but not over properties. That is, while we can state something like

There exists x such that x is a cube.

we cannot quantify over a predicate. That is, the statement

There exists a property P such that a cube satisfies P.

This statement does not make sense in first-order logic, but makes sense in second-order logic.

[a]This is true if we look at the contrapositive: $x \equiv 1 \implies x^2 \equiv 1$.