Representation Theory

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1 Introduction

Say that we have some abstract group G. Groups are defined purely by how elements multiply, which can make them hard to get a grip on. Linear algebra, on the other hand, is something we understand extremely well. So here is the idea of representation theory: *study a group by letting it act linearly on a vector space.* Each group element becomes a matrix, group multiplication becomes matrix multiplication, and then we can bring in all of the tools of linear algebra (eigenvalues, traces, decompositions) to learn about G.

Therefore, the main questions we will ask are the following.

1. What are all the ways that G can act linearly on a vector space?
2. Can we break each such action into smaller, irreducible pieces, like prime factorization?
3. How can we tell two representations apart? (This will be answered with characters.)

Unless stated otherwise, all vector spaces are finite-dimensional over $\mathbb{C}$ and all groups are finite. Lie groups and Lie algebras are defined in my notes on differential geometry, and we will refer to them there.

2 Representations of Groups

We will assume that V is a finite-dimensional vector space over the field $\mathbb{C}$, and we recall the general linear group $\mathrm{GL}(V)$ of automorphisms of V.

Definition 2.1 (Representation)

A **representation** of an (algebraic) group $\mathcal{G}$ is a homomorphism

$$\rho : G \to \mathrm{GL}(V) \tag{1}$$

for some vector space V. That is, given an element $g \in \mathcal{G}$, $\rho(g) \in \mathrm{GL}(V)$, meaning that $\rho(g)(v) \in V$. Additionally, since it is a homomorphism, the algebraic structure is preserved.

$$\rho(g_1 \cdot g_2) = \rho(g_1) \cdot \rho(g_2) \tag{2}$$

where $\cdot$ on the left hand side is the abstract group multiplication while the $\cdot$ on the right hand side is matrix multiplication. To shorten the notation, we will denote

$$gv = \rho(g)v, \ v \in V \tag{3}$$

Since ρ is a group morphism, we have

$$g_2(g_1 v) = (g_2 g_1)v \iff \rho(g_2)\big(\rho(g_1)(v)\big) = \big(\rho(g_2)\rho(g_1)\big)(v) \tag{4}$$

Additionally, since g (that is, $\rho(g)$) is a linear map,

$$g(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 g v_1 + \lambda_2 g v_2 \tag{5}$$

Usually, we refer to the map as the representation, but if the map is well-understood, we just call the vector space V the representation and say that the group acts on this vector space.

Example 2.1 (Representations of $\mathrm{GL}(2, \mathbb{C})$)

The group $\mathrm{GL}(2, \mathbb{C})$ can be represented a by the vector space $\mathbb{C}^2$, or explicitly, by the group of 2×2 matrices over $\mathbb{C}$ with nonzero determinant.

$$\mathrm{GL}(2, \mathbb{C}) \overset{id}{\longmapsto} \mathrm{Mat}(2, \mathbb{C}) \tag{6}$$

This is a trivial representation.

We now show a nontrivial representation of $\mathrm{GL}(2, \mathbb{C})$.

Example 2.2 (The Second Symmetric Power $\mathrm{Sym}^2\mathbb{C}^2$)

We take $\mathrm{Sym}^2\mathbb{C}^2$, the second symmetric power of $\mathbb{C}^2$. Note that given a basis $x_1, x_2 \in \mathbb{C}^2$, the set

$$\{x_1 \odot x_1, x_1 \odot x_2, x_2 \odot x_2\} \tag{7}$$

forms a basis of $\mathrm{Sym}^2\mathbb{C}^2 \implies \dim \mathrm{Sym}^2\mathbb{C}^2 = 3$. So, we want to represent $\mathrm{GL}(2, \mathbb{C})$ by associating its element with elements of $\mathrm{GL}(Sym^2\mathbb{C}^2)$. More concretely, we are choosing to represent a 2×2 matrix

over $\mathbb{C}$ with a 3×3 matrix group (since $\text{GL}(Sym^2\mathbb{C}^2) \simeq \text{GL}(3, \mathbb{C})$. Clearly,

$$\begin{aligned}\rho(g)(x_1 \odot x_1) &= g(x_1) \odot g(x_1) \in Sym^2\mathbb{C}^2 \\ \rho(g)(x_1 \odot x_2) &= g(x_1) \odot g(x_2) \\ \rho(g)(x_2 \odot x_2) &= g(x_2) \odot g(x_2)\end{aligned}$$

To present this in matrix form, let us have an element in $\text{GL}(2, \mathbb{C})$

$$\mathcal{A} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{8}$$

We evaluate the corresponding representation in $\text{GL}(Sym^2\mathbb{C}^2)$. Using the identities above, we have

$$\begin{aligned}\rho(g)(x_1 \odot x_1) &= g(x_1) \odot g(x_1) \\ &= (ax_1 + cx_2) \odot (ax_1 + cx_2) \\ &= a^2 x_1 \odot x_1 + 2acx_1 \odot x_2 + c^2 x_2 \odot x_2 \\ \rho(g)(x_1 \odot x_2) &= g(x_1) \odot g(x_2) \\ &= (ax_1 + cx_2) \odot (bx_1 + dx_2) \\ &= abx_1 \odot x_1 + (ad + bc)x_1 \odot x_2 + cdx_2 \odot x_2 \\ \rho(g)(x_2 \odot x_2) &= g(x_2) \odot g(x_2) \\ &= (bx_1 + dx_2) \odot (bx_1 + dx_2) \\ &= b^2 x_1 \odot x_1 + 2bdx_1 \odot x_2 + d^2 x_2 \odot x_2\end{aligned}$$

And this completely determines the matrix. So,

$$\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix} \tag{9}$$

is the 3×3 representation of $\mathcal{A}$ in $\text{GL}(Sym^2\mathbb{C}^2)$.

We continue to define maps between two representations of $\mathcal{G}$.

Definition 2.2 (Morphism of Representations)

A **morphism** between 2 representations

$$\begin{aligned}\rho_1 &: \mathcal{G} \to \text{GL}(V_1) \\ \rho_2 &: \mathcal{G} \to \text{GL}(V_2)\end{aligned}$$

of some group but not necessarily the same vector space is a linear map $f : V_1 \to V_2$ that is **compatible** with the group action. That is, f satisfies the property that for all $g \in \mathcal{G}$

$$f \circ g = g \circ f \tag{10}$$

Again, we use the shorthand notation that $g = \rho(g)$, meaning that the statement above really translates to $f \circ \rho(g) = \rho(g) \circ f$. This is equivalent to saying that the following diagram commutes.

$$\begin{array}{ccc} V_1 & \xrightarrow{\rho_1(g)} & V_1 \\ \downarrow f & & \downarrow f \\ V_2 & \xrightarrow{\rho_2(g)} & V_2 \end{array} \tag{11}$$

Definition 2.3 (Subrepresentation)

Let V be a representation of $\mathcal{G}$. A **subrepresentation** is a subspace $W \subset V$ such that for all $g \in \mathcal{G}$ and for all $w \in W$,

$$\rho(g)(w) \in W \tag{12}$$

Example 2.3 (Trivial Subrepresentations)

V and $\{0\}$ are always subrepresentations of V.

We now introduce the "building blocks" of all representations.

Definition 2.4 (Irreducible Representation)

A representation W is **irreducible representation** if $\{0\}$ and W are the only subrepresentations of W.

Definition 2.5 (Direct Sum of Representations)

The direct sum of representations is a representation. That is, if U is a representation and V is a representation, then $U \oplus V$ is a representation. That is, if

$$\rho_1 : \mathcal{G} \to U,\ \rho_1(g) = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \tag{13}$$

and

$$\rho_2 : \mathcal{G} \to V,\ \rho_2(g) = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} \tag{14}$$

are two representations of the same group element $g \in \mathcal{G}$, then

$$(\rho_1 \oplus \rho_2) : \mathcal{G} \to (U \oplus V),\ (\rho_1 \oplus \rho_2)(g) = \begin{pmatrix} u_1 & u_2 & 0 & 0 \\ u_3 & u_4 & 0 & 0 \\ 0 & 0 & v_1 & v_2 \\ 0 & 0 & v_3 & v_4 \end{pmatrix} \tag{15}$$

is a bigger representation of g in $U \oplus V$.

Lemma 2.1 (Schur's Lemma)

Let V_1, V_2 be irreducible representations and let $f : V_1 \to V_2$ be a morphism (of representations). Then, either

1. f is an isomorphism.
2. $f = 0$

Furthermore, any 2 isomorphisms differ by a constant. That is,

$$f_1 = \lambda f_2 \tag{16}$$

Proof. $\ker f$ is clearly a vector space. Furthermore, it is a subrepresentation (since it is a subspace of V_1) $\implies$ $\ker f = V$ or $\ker f = 0$. If $\ker f = V$, then $f = 0$ and the theorem is satisfied. If $\ker f = 0$,

then f is injective, and $\text{Im}\, f$ is a subrepresentation of $V_2 \implies \text{Im}\, f = 0$ or $\text{Im}\, f = V_2$. But $\text{Im}\, f \neq 0$ since f is injective, so $\text{Im}\, f = V_2 \implies f$ is surjective $\implies f$ is bijective, that is, f is an isomorphism of vector spaces. So, the inverse f^{-1} exists, and this map f^{-1} satisfies

$$f^{-1} \circ \rho_2(g) = \rho_1(g) \circ f^{-1} \tag{17}$$

To prove the second part, without loss of generality, assume that the first isomorphism is the identity mapping. That is,

$$f_1 = id \tag{18}$$

Since we are working over the field $\mathbb{C}$, we can find an eigenvector of f_2. That is, there exists a $v \in V_1$ such that

$$f_2(v) = \lambda v \tag{19}$$

Now, we define the map

$$f : V_1 \to V_2,\ f \equiv f_2 - \lambda f_1 \tag{20}$$

Clearly, $\ker f \neq 0$, since $v \in \ker f$. That is, we have a map f between 2 irreducible representations that has a nontrivial kernel. This means that $f = 0 \implies f_2 = \lambda f_1$.

Theorem 2.2 (Maschke's Theorem)

Let V be finite dimensional, with $\mathcal{G}$ a finite group. Then, V can be decomposed as

$$V = \bigoplus_i V_i \tag{21}$$

where each V_i is an irreducible representation of $\mathcal{G}$.

Proof. By induction on dimension, it suffices to prove that if W is a subrepresentation of V, then there exists a subrepresentation $W' \subset V$ such that $W \oplus W' = V$. So, if V isn't an irreducible representation, it can always be decomposed into smaller subrepresentations W and W' that direct sum to V. Now, we define the canonical (linear) projection

$$\pi : V \to W \tag{22}$$

Then, we define the new map

$$\tilde{\pi} : V \to W,\ \tilde{\pi}(v) \equiv \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \rho(g)\big|_W \circ \pi \circ \rho(g)^{-1} \tag{23}$$

This "averaging" of the group elements are done so that this mapping is a map of representations. This implies that

$$V = W \oplus \ker \tilde{\pi} \tag{24}$$

meaning that V can indeed be decomposed into direct sums of subrepresentations.

3 Character Theory

Definition 3.1 (Character)

The **character** of a representation (V, ρ) is the function $\chi_V : G \to \mathbb{C}$ given by

$$\chi_V(g) \coloneqq \operatorname{Tr}\big(\rho(g)\big) \tag{25}$$

Characters are constant on conjugacy classes, since the trace is invariant under conjugation. TBD: orthogonality relations, character tables.

4 Representations of Lie Groups and Lie Algebras

Let $\mathcal{G}$ be a Lie group and let

$$\rho : \mathcal{G} \to \mathrm{GL}(V) \tag{26}$$

be the representation of $\mathcal{G}$. Then, let $\mathfrak{g}$ be the Lie algebra of $\mathcal{G}$, and $\mathfrak{gl}(V)$ be the Lie algebra of $\mathrm{GL}(V)$. Then, ρ induces another homomorphism

$$\varrho : \mathfrak{g} \to \mathfrak{gl}(V) \tag{27}$$

where the bracket structure (in this case, the comutator in the matrix algebra) is preserved.

$$\varrho([X, Y]) = [\varrho(X), \varrho(Y)] \tag{28}$$

We can visualize this induced homomorphism with the following commutative diagram, which states that $\rho \circ \exp = \exp \circ \varrho$.

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\rho} & \mathrm{GL}(V) \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow{\varrho} & \mathfrak{gl}(V) \end{array} \tag{29}$$

Note that there are very crucial differences between ρ and ϱ. First, ρ is a homomorphism between **groups**, while ϱ is a homomorphism between **vector spaces**. Additionally, $\mathrm{GL}(V)$ is a group, not a linear space, while $\mathfrak{gl}(V)$ is a linear space. Finally, note that $\mathrm{GL}(V)$ is restricted to only matrices with nonzero determinants, while the elements of $\mathfrak{gl}(V)$ can be any matrix.

Example 4.1 (Matrix Representation of $SE(n)$)

The representation of $\mathrm{SE}(n)$ to $\mathrm{GL}(n+1, \mathbb{R})$ and $\mathfrak{se}(n)$ to $\mathfrak{gl}_{n+1}\mathbb{R}$ induces the second homomorphism $\varrho : \mathfrak{gl}_{n+1}\mathbb{R} \to \mathrm{GL}(n+1, \mathbb{R})$.

For our case, we will consider that any representation can be written as a direct sum of irreducible representations. We will now proceed to find an irreducible representation of $\mathfrak{sl}_2\mathbb{C}$. This means that we want to find the smallest (lowest dimensional) vector space V such that there exists a representation

$$\varrho : \mathfrak{sl}_2\mathbb{C} \to \mathfrak{gl}(V) \tag{30}$$

We will write, as shorthand notation, that

$$H = \varrho(H), X = \varrho(X), Y = \varrho(Y) \tag{31}$$

Clearly, $H, X, Y \in \mathfrak{gl}(V) \simeq \mathfrak{gl}(\mathbb{C}^n)$. By the spectral theorem, we can find an orthonormal basis of eigenvectors $e_1, e_2, ..., e_n$ of the mapping H such that

$$He_i = \lambda_i e_i, \ \lambda_i \in \mathbb{C} \tag{32}$$

Since $[H, X] = 2X$, it follows that $HXe_i - XHe_i = 2Xe_i \implies H(Xe_i) = (\lambda_i + 2)(Xe_i) \implies Xe_i$ for all $i = 1, 2, ..., n$ are also eigenvectors of H with eigenvalue $(\lambda_i + 2)$, or $Xe_i = 0$. So, X is a "ladder operator" that maps each eigenvector e_i with eigenvalue λ_i to a different eigenvector e_j with eigenvalue $\lambda_j = \lambda_i + 2$. Having nowhere to be mapped to, the eigenvector with the largest eigenvalue (which must exist since V is finite dimensional) will get mapped to the 0 vector by X. Let us denote this eigenvector having the maximum eigenvalue m, as v_m.

Similarly, $[H, Y] = -2Y$ implies that

$$HYe_i - YHe_i = -2Ye_i \implies H(Ye_i) = (\lambda_i - 2)(Ye_i) \tag{33}$$

implying that Y maps each eigenvector e_i with eigenvaue λ_i to another eigenvector e_j with eigenvalue $\lambda_j = \lambda_i - 2$, except for the eigenvector with smallest eigenvalue, which gets mapped to 0. Since Y clearly maps each eigenvector to a different eigenvector that has a strictly decreasing eigenvalue, we can construct a basis of V to be

$$\{v_m, Yv_m, Y^2v_m, Y^3v_m, ..., Y^{n-1}v_m\} \tag{34}$$

(remember that $Y^n v_m = 0$). So, elements of $\mathfrak{sl}_2\mathbb{C}$ acts on the space V with basis above. To continue, we introduce the following theorem.

Theorem 4.1 (Action of the Raising and Lowering Operators)

$$XY^j v_m = j(m - j + 1)Y^{j-1}v_m \tag{35}$$

Proof. By induction on j using bracket relations.

V is n-dimensional. Since $Y^n v_m = 0$ and $Y^{n-1}v_m \neq 0$, we use the theorem above to get

$$0 = XY^n v_m = n(m - n + 1)Y^{n-1}v_m \implies m - n + 1 = 0 \tag{36}$$

So, $n = m + 1$, which means that the eigenvalues of H are

$$m, m-2, m-4, \dots, m-2(n-1) = -m \tag{37}$$

and we are done. We now classify the 1, 2, and 3 dimensional irreducible representations of $\mathfrak{sl}_2\mathbb{C}$.

1. When $n = 1$ (i.e. dimension is 1), $m = n - 1 = 0$, meaning that the greatest (and only) eigenvalue is 0. That is,
$$Hv_0 = 0,\ Xv_0 = 0,\ Yv_0 = 0 \tag{38}$$
which is the trivial representation of $\mathfrak{sl}_2\mathbb{C}$. Explicitly, we can completely define the representation (which is a linear homomorphism) with the three equations.
$$\varrho(H) = (0),\ \varrho(X) = (0),\ \varrho(Y) = (0) \tag{39}$$

2. When $n = 2$ and $m = 1$. We now look for a 2 dimensional irreducible representation. The eigenvalues are 1 and -1, with $\{v_1, v_{-1}\}$ as a basis of 2 dimensional space V. Then we have
$$\begin{aligned} Hv_1 = v_1,\ Hv_{-1} = -v_{-1} \\ Xv_1 = 0,\ Xv_{-1} = v_1 \\ Yv_1 = v_{-1},\ Yv_{-1} = 0 \end{aligned}$$
which explicitly translates to the representation ϱ being defined
$$\varrho(H) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{40}$$

3. When $n = 3 \implies m = 2$, the basis is $\{v_{-2}, v_0, v_2\}$ with eigenvalues $2, 0, -2$, and the irreducible representation ϱ is defined
$$\varrho(H) = \begin{pmatrix} 2 & & \\ & 0 & \\ & & -2 \end{pmatrix}, \varrho(Y) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \varrho(X) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{41}$$

4. The same process continues on for $n = 4, 5, ...$, and this entirely classifies the irreducible representations of $\mathfrak{sl}_2\mathbb{C}$.

The adjoint maps Ad and ad defined in the notes on differential geometry are also examples of representations: Ad is a representation of $\mathcal{G}$ on its Lie algebra $\mathfrak{g}$, and ad is the induced representation of $\mathfrak{g}$ on itself.

5 Tensor Products of Representations

Definition 5.1 (Tensor Product of Representations)

If V and W are two different representations of a group $\mathcal{G}$, then we know that $V \oplus W$ is also a representation of $\mathcal{G}$. Furthermore, the tensor product space $V \otimes W$ also defines a representation of $\mathcal{G}$. That is, given representations

$$\rho_V : \mathcal{G} \to \mathrm{GL}(V)$$
$$\rho_W : \mathcal{G} \to \mathrm{GL}(W)$$

The homomorphism $\rho_V \otimes \rho_W : \mathcal{G} \to \mathrm{GL}(V \otimes W)$ is also a representation of $\mathcal{G}$, which is defined

$$(\rho_V \otimes \rho_W)(g)(v \otimes w) \equiv \rho_V(g)(v) \otimes \rho_W(g)(w) \tag{42}$$

or represented in shorthand notation,

$$g(v \otimes w) \equiv (gv) \otimes (gw) \tag{43}$$

We know that $\exp(H)$ acts on V and W since it is an element of $\mathrm{GL}(V)$ and $\mathrm{GL}(W)$. This means that

$$\exp(H)(v \otimes w) \equiv \big(\exp(H)(v)\big) \otimes \big(\exp(H)(w)\big) \tag{44}$$

If H ($= \rho_V(H)$ or $\rho_W(H)$) has an eigenvalue λ on v in V and eigenvalue μ on w in W, then

$$\exp(H)(v \otimes w) = (e^{\lambda} v) \otimes (e^{\mu} w) = e^{\lambda+\mu} v \otimes w \tag{45}$$

That is, eigenvalues of H **add** on tensor products.

Example 5.1 (Weights of the 2-Dimensional Representation)

Recall that the 2 dimensional representation V of $\mathfrak{sl}_2\mathbb{C}$ has eigenvalues 1 and -1 (with corresponding eigenvectors e_1 and e_{-1}). So, $V \otimes V$ has eigenvalues

$$(-1) + (-1) = -2, \;\; (-1) + 1 = 0$$
$$1 + (-1) = 0, \;\; 1 + 1 = 2$$

Therefore, the eigenvalues of $V \otimes V$ is -2 (geometric multiplicity of 1), 0 (geometric multiplicity of 2), and 2 (geometric multiplicity of 1), (Notation-wise, the n-dimensional irreducible representation of $\mathfrak{sl}_2\mathbb{C}$ is denoted $\mathbf{n}$.) which means that

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1} \tag{46}$$

We can decompose $V \otimes V$ into its symmetric and exterior power components. $\mathrm{Sym}^2 V$ has basis (of eigenvectors)

$$\{e_{-1} \odot e_{-1},\; e_{-1} \odot e_1,\; e_1 \odot e_1\} \tag{47}$$

where the corresponding eigenvalues are -2, 0, and 2, respectively. So, $\dim Sym^2 V = 3$, which means that $Sym^2 V = \mathbf{3}$. As for the exterior power component of V, $\Lambda^2 V$ has basis $\{e_{-1} \wedge e_1\}$ with eigenvalue $= 0 \implies \dim \Lambda^2 V = 1$, meaning that $\Lambda^2 V = \mathbf{1}$. Therefore,

$$V \otimes V = Sym^2 V \oplus \Lambda^2 V = \mathbf{3} \oplus \mathbf{1} \tag{48}$$