Duke University
Department of Physics

Physics

Personal Notes

Muchang Bahng

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Email any inquiries or comments to muchang.bahng@duke.edu.

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Chapter 1

Introduction

This book is a series of notes from physics courses that I have independently studied. As the reader will see in the table of contents, this book covers a variety of topics in mainly undergraduate and occasionally graduate level physics. Additionally, I have ordered the chapters in such a way that prerequisite information for future chapters is initially covered, but the content in these courses are so interdependent that it made it difficult to do so completely.

This is book is not too rigorous nor too non-rigorous on its introduction to the topics mentioned. Unlike most textbooks, this book does not focus on a specific field of physics; it provides an introduction to a wide variety of fields. This book is mainly aimed for people who would like to have a non-rigorous introduction to the courses covered and to students who have taken these courses and would like to review them briefly. I believe that this book serves as an excellent glossary that comprehensively covers the important, fundamental ideas in the courses. Furthermore, I have tried to place an emphasis on the geometric interpretations behind many of the concepts explained in this book.

Finally, I would like to state that this book is a work in progress, and I welcome any constructive criticisms that readers may have for this book. Any questions and inquiries can be emailed to: muchang.bahng@duke.edu. I would like to thank the professors, peers, and textbooks that have helped me understand the material in this book. I would also like to extend my gratitude to those that may help me in the future.

Chapter 2

Classical Mechanics

We will usually work in the vector space $\mathbb{R}^3$.

2.1 Elementary Principles

Definition 2.1.1 (Displacement Vector)**.** We denote $\mathbf{r}$ as the displacement vector of a particle from given origin O. We can represent $\mathbf{r}$ in the following ways:

$$\begin{aligned}\mathbf{r} &= (x, y, z) \\ &= r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + r_3\mathbf{e}_3 \\ &= r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k} \\ &= x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}\end{aligned}$$

We may use any of these notations from now on. Furthermore, $\mathbf{r}$ can be interpreted as a path function mapping from a time interval in the time continuum $\mathbb{R}^3$ to $\mathbb{R}^3$.

$$\mathbf{r}(t) = \big(x(t), y(t), z(t)\big)$$

This function models the movement of the particle through $\mathbb{R}^3$.

Definition 2.1.2 (Velocity, Acceleration Vector)**.** Given the displacement vector $\mathbf{r}$ of a particle, we define its velocity vector (function) as

$$\mathbf{v} \equiv \dot{\mathbf{r}} \equiv \frac{d\mathbf{r}}{dt} \equiv \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

and its acceleration vector (funtion) as

$$\mathbf{a} \equiv \frac{d\mathbf{v}}{dt} \equiv \left(\frac{dv_x}{dt}, \frac{dv_y}{dt}, \frac{dv_z}{dt}\right)$$

Clearly $\mathbf{a} = \ddot{\mathbf{r}}$.

Definition 2.1.3 (Mass)**.**

Definition 2.1.4 (Linear Momentum)**.** The *linear momentum* $\mathbf{p}$ of a particle is defined as the product of its mass and velocity.

$$\mathbf{p} = m\mathbf{v} = m\frac{d\mathbf{r}}{dt}$$

Definition 2.1.5 (Force)**.**

2.1.1 Frames

The motion of a body can only be described relative to something else. We can identify this object with spatial coordinates and call it the origin.

Definition 2.1.6 (Spatial, Temporal Reference Frames)**.** When dealing with problems in classical mechanics, we would like to model properties of particles in $\mathbb{R}^3$. But before we do that, we must determine the *reference frame* of $\mathbb{R}^3$; that is, we must (implicitly or explicitly) specify

1. a choice of spatial origin O (which can be in motion), and
2. a choice of basis (i.e. axes in $\mathbb{R}^3$)

In the one-dimensional temporal axes, we must choose the origin in time

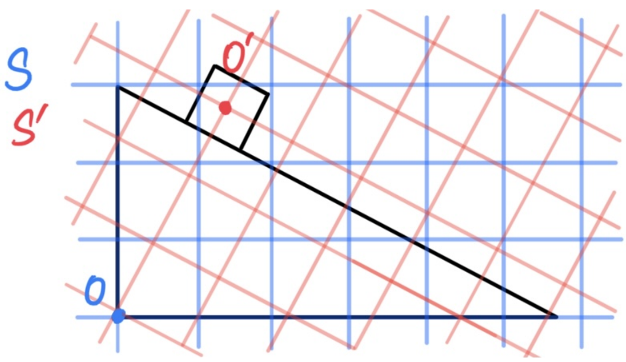
1. $t = t_0$

Therefore, when determining a reference frame, we must identify the spatial origin, the spatial axes, and the temporal origin.

We may be more comfortable with orthonormal reference frames with a stationary origin, but sometimes, problems may be greatly simplified with more complex frames. We list some examples of frames.

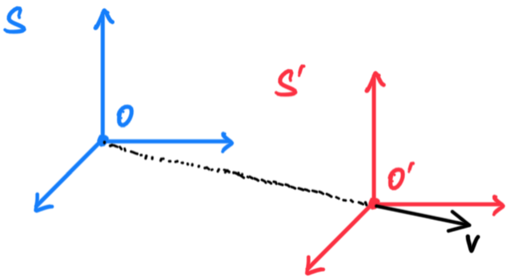
Example 2.1.1 (Common Frames)**.** *We begin with some simple frames, and transition to more complex frames. The origin will be labeled with O, and for simplicity of visualizing, we may restrict ourselves to $\mathbb{R}^2$.*

1. *Let $\mathcal{S}$ be the normal orthonormal frame of a system consisting of a block sliding down from a ramp beginning at $t = 0$. For modeling a point (representing the center of mass of the block) sliding down the ramp, we can conveniently use frame $\mathcal{S}'$. Notice that, relative to $\mathcal{S}$, the origin and orientation of $\mathcal{S}'$ have changed but are fixed.*

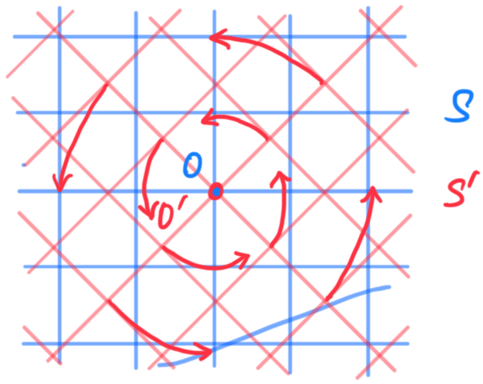


If we wish to change the temporal frame, we can let the block slide initially at $t = t_0 \neq 0$.

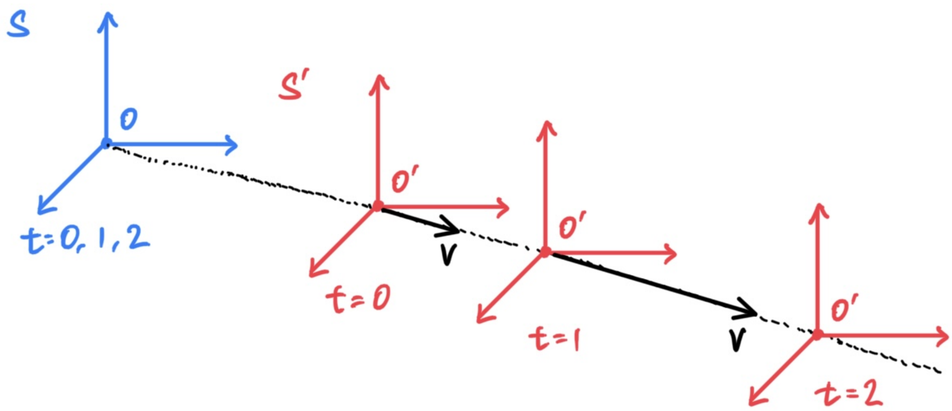
2. *We can represent two frames $\mathcal{S}, \mathcal{S}'$ in relative motion at a constant velocity. Relative to $\mathcal{S}$, the origin of $\mathcal{S}'$ is changing (with constant velocity v) but the orientation has not changed.*



3. *Given frame $\mathcal{S}$, the frame $\mathcal{S}'$ rotates counterclockwise around the origin at a constant rate. Relative to $\mathcal{S}$, the origin of $\mathcal{S}'$ has not changed but the orientation is changing.*

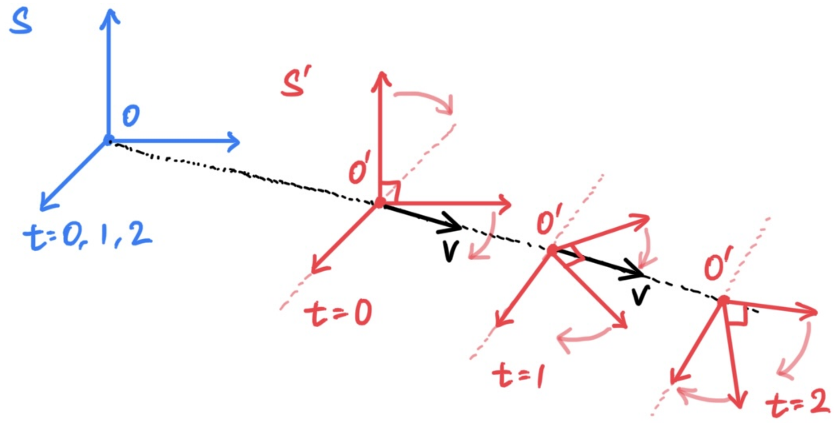


4. *We can represent two frames $\mathcal{S}, \mathcal{S}'$ in relative motion that is also accelerating. Relative to $\mathcal{S}$, the origin of $\mathcal{S}'$ is changing but the orientation has not changed.*



With the additional transformation $t \mapsto t - t_0$, we get a temporal change in frame.

5. *Given frame $\mathcal{S}$, the frame $\mathcal{S}'$ is in constant relative motion that is also rotating. Relative to $\mathcal{S}$, both the origin and orientation of $\mathcal{S}'$ is changing.*



Definition 2.1.7 (Newtonian Inertial Frame)**.** An inertial frame of reference is a frame of reference in which Newton's first law is satisfied. In other words, it is a frame of reference in which every particle with no force acting upon it is traveling in constant velocity. This means that given an inertial frame $\mathcal{S}$, an inertial frame $\mathcal{S}'$ is nonaccelerating and nonrotating relative to $\mathcal{S}$.

Theorem 2.1.1 (Transformations Among Newtonian Inertial Frames). Newtonian inertial frames transform among each other according to the Galilean group of symmetries, defined

$$\text{Tran}\mathbb{R}^3 \equiv \mathbb{R}^4 \times H \times \text{O}(3)$$

where H is the group of transformations of the form

$$(x, y, z, t) \longmapsto (x + at, y + bt, z + ct, t)$$

This makes sense with what we have described so far:

1. $\mathbb{R}^4$ represents the shift in the $3 + 1$ spatial and temporal origin of the new frame
2. H represents the constant velocity motion of the origin of the new frame
3. $O(3)$, the orthogonal group, represents the change in orientation of the new frame (which is fixed and not changing at time passes).

2.1.2 Newton's Laws of Motion

Definition 2.1.8 (Newton's Second Law)**.** In an inertial frame of reference, the force $\mathbf{F}$ of a particle has the property:

$$\mathbf{F} \equiv \frac{d\mathbf{p}}{dt} \equiv \dot{\mathbf{p}}$$

Clearly, by definition, this definition is equivalent to the notion that $\mathbf{F} = m\mathbf{a}$. Furthermore, we can see that by the fundamental theorem of calculus, the change in momentum of a particle from time $t = t_1$ to $t = t_2$ can be computed by integrating the force applied to it (which may change over time) over interval $[t_1, t_2]$.

$$\Delta\mathbf{p} = \int_{t_1}^{t_2} \mathbf{F}(t)\, dt$$

$\Delta\mathbf{p}$ is also called the *impulse*, denoted $\mathbf{J}$.

Corollary 2.1.1.1 (Newton's First Law). In an inertial frame of reference, a particle moves with constant velocity $\mathbf{v}$ if no force is applied to it.

$$\mathbf{F} = 0 \implies \mathbf{a} = 0$$

Clearly, given a force $\mathbf{F}$ acting on a point particle Newton's second law gives us a second-order vector-valued differential equation of the form

$$\mathbf{F} = m\ddot{\mathbf{r}}$$

Lemma 2.1.2 (Equation of Motion for a Particle with Constant Force). Given a constant force $\mathbf{F_0}$ acting on a particle in $\mathbb{R}^3$ with a reference frame, let $\mathbf{r}(t)$ be the displacement vector of the particle as time t passes. Then, the equation of motion for the particle is

$$\mathbf{r}(t) = \mathbf{x_0} + \mathbf{v_0}t + \frac{1}{2m}\mathbf{F_0}t^2 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \begin{pmatrix} v_{x0} \\ v_{y0} \\ v_{z0} \end{pmatrix} t + \frac{1}{2m} \begin{pmatrix} F_{0x} \\ F_{0y} \\ F_{0z} \end{pmatrix} t^2$$

where $\mathbf{x_0}$ is the initial displacement and $\mathbf{v_0}$ is the initial velocity. Note that we need two vectors to represent the initial condition since this second order differential equation can be equivalently represented as a system of six first-order differential equations, which has a 6-dimensional phase space.

Proof. We integrate to get the equation for velocity

$$\dot{\mathbf{r}}(t) = \int \ddot{\mathbf{r}}(t)\, dt = \mathbf{v_0} + \frac{1}{m}\mathbf{F_0}t$$

and integrate again to get the equation for displacement

$$\mathbf{r}(t) = \int \dot{\mathbf{r}}(t)\, dt = \mathbf{r_0} + \mathbf{v_0}t + \frac{1}{2m}\mathbf{F_0}t^2$$

■

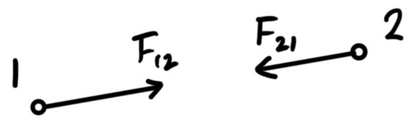
Evidently, Newton's two laws hold only in the special, inertial reference frames. Usually, we use the first law to find whether a reference frame $\mathcal{S}$ is inertial and after, we can claim as an experimental fact that the second law holds in these same inertial frames.

Example 2.1.2 (Reference Frame Fixed to the Earth). *One important fact to realize is that even though we may think that a reference frame fixed to the Earth is inertial, it actually is not: it is only an approximation of an inertial frame. This is because the Earth rotates on its axis once a day, circles around the sun once a year, and the sun orbits the Milky Way galaxy.*

Although these effects are very small, there are several phenomena (tides and trajectories of long-range projectiles) that are most simply explained by taking into account the noninertial character of a frame fixed to the Earth.

Theorem 2.1.3 (Newton's Third Law of Motion). If object 1 exerts a force $\mathbf{F}_{21}$ on object 2, then object 2 always exerts a reaction force $\mathbf{F}_{12}$ on object 1 given by

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$



It turns out that this third law brings forth the law of conservation of momentum, as we will see now.

2.1.3 Systems of Particles and Total Linear Momentum

A *physical system* is a portion of the physical universe chosen for analysis. Everything outside the system is known as the *environment*, which is ignored except for its effects on the system. An isolated system is one that has negligible interaction with its environment, but external forces may act on the system.

Definition 2.1.9 (Center of Mass of a System of Particles)**.** Given a system of n particles in a reference frame of $\mathbb{R}^3$ with displacement vectors

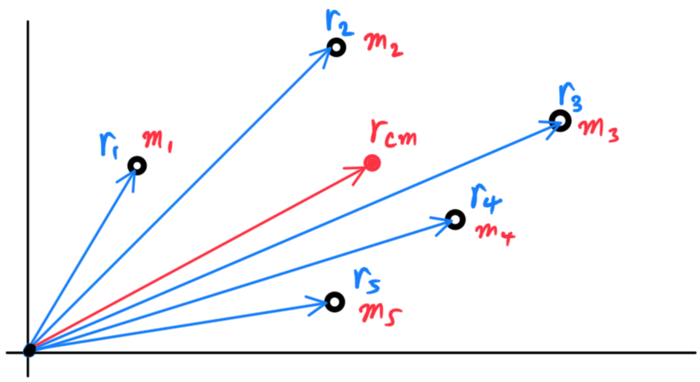
$$\mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_n}$$

and masses

$$m_1, m_2, \ldots, m_n$$

its *center of mass* is defined by the weighed sum of their positions:

$$\boldsymbol{R} \equiv \mathbf{r}_{cm} \equiv \frac{m_1\mathbf{r_1} + m_2\mathbf{r_2} + \ldots + m_n\mathbf{r_n}}{m_1 + m_2 + \ldots + m_n} \equiv \frac{\sum_i m_i\mathbf{r_i}}{\sum_i m_i}$$



If we wish the view the entire system as a point, then we can view it as a point with displacement vector $\mathbf{R}$ and mass $m = \sum_i m_i$.

Definition 2.1.10 (Center of Mass for Continuously Distributed Body)**.** The center of mass of a body that is distributed continuously is

$$\boldsymbol{R} \equiv = \frac{1}{M}\int_V \varrho \boldsymbol{r}\, dV$$

where the integral runs over the volume V, dV denotes an element of volume, and ϱ is the the mass density of the body.

Definition 2.1.11 (Total Linear Velocity, Total Linear Momentum)**.** Since we have defined the displacement vector of the center of mass of the system, we can differentiate it with respect to t to get

$$\mathbf{V} \equiv \frac{\sum_i m_i \mathbf{v_i}}{\sum_i m_i}$$

But we know that $m = \sum_i m_i$, so this means that we can define the total momentum as

$$\begin{aligned} \mathbf{P} &\equiv \left(\sum_i m_i\right)\left(\frac{\sum_i m_i \mathbf{v_i}}{\sum_i m_i}\right) \equiv m\mathbf{V} \\ &\equiv \sum_i \mathbf{p_i} \end{aligned}$$

Note that this allows us to interpret the linear momentum of a system in two equivalent ways:

1. The total linear momentum is just the sum of the individual linear momenta of each particle.
2. Let the center of mass of the system be $\boldsymbol{R}$. Then the rate at which this center of mass point moves is $\boldsymbol{V}$, and multiplying this by the mass of the system gives the total momentum.

Definition 2.1.12 (Total Force)**.** Deriving the equation of the total momentum yields the *total force* of the system.

$$\begin{aligned} \mathbf{F_{cm}} &\equiv m\mathbf{a_{cm}} \\ &\equiv \sum_i \mathbf{F_i} \end{aligned}$$

Again, we can interpret the total force as the sum of all the individual forces on each particle, and this sum represents the force (change in momentum) of the center of mass of the system.

Law of Conservation of Momentum

Let us have a system of n particles, with each particle labeled $1, 2, \ldots, n$, and let the external force (outside the system) acting on particle i be labeled

$$\mathbf{F_i^{ext}}$$

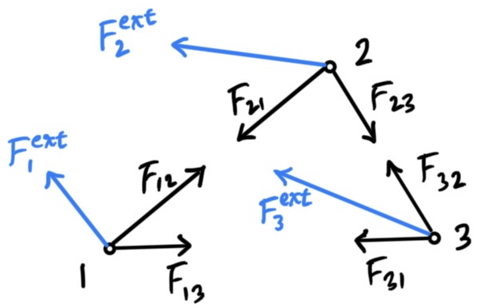
Let the force of particle j acting on object i be represented as

$$\mathbf{F_{ij}}$$

and finally let the net force on particle i be represented with

$$\mathbf{F_i} \equiv \mathbf{F_i^{ext}} + \sum_{j \neq i} \mathbf{F_{ij}}$$

A simple diagram is shown when $n = 3$.



Taking the total force of this system results in

$$\sum_i \mathbf{F_i} = \sum_i \mathbf{F_i^{ext}} + \sum_i \sum_{j \neq i} \mathbf{F_{ij}}$$
$$= \sum_i \mathbf{F_i^{ext}}$$

by Newton's third law that $\mathbf{F_{ij}} = -\mathbf{F_{ji}}$. This leads to a very important realization.

Lemma 2.1.4. Internal forces within a system do not affect the total force of the system.

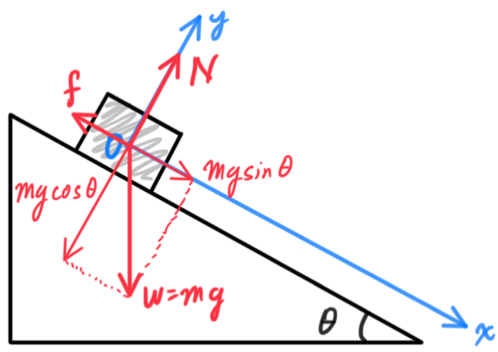
Theorem 2.1.5 (Law of Conservation of Momentum). Interpreting the total force of the system $\mathbf{F}$ as the derivative of the momentum $\mathbf{P}$, we get

$$\dot{\mathbf{P}} = \sum_i \mathbf{F_i^{ext}}$$

This means that if there is no external force acting on a system, then its momentum must be constant. This can also be equivalently expressed as

$$\boldsymbol{F^{ext}} = M\ddot{\boldsymbol{R}}$$

Example 2.1.3 (Block Sliding Down an Incline). *A box of mass m is observed accelerating from rest down an include that has coefficient of friction μ and is at angle θ from the horizontal. How far will it travel in time t? Sketching it and drawing a free body diagram gives*



We choose our frame of reference by selecting the origin to be at the center of mass of the block, the x-axis point down the slope, y-axis normal to the slope, and z-axis pointing *out of the page. Doing so clearly simplifies the problem since we can observe that only the x-component of the displacement vector is changing.*

1. *There are no forces in the z direction, so $F_z = m\ddot{z} = 0$. Solving the differential equation gives*
$$z = z_0 + v_{z0}t$$
but since the initial displacement and velocities $z_0, v_{z0} = 0$, $z = 0$.

2. *There is no motion in the y-direction, so $\ddot{y} = 0 \implies F_y = 0$. Solving the initial value problem also gives $y = 0$. Furthermore, with a bit of trigonometry, we can see that this implies that*
$$F_y = N - mg\cos\theta = 0 \implies N = mg\cos\theta$$
and therefore, the frictional force is $f = \mu mg\cos\theta$.

3. *Finally, we can see that the x-component of the weight force is $w_x = mg\sin\theta$, and so*
$$F_x = w_x - f = mg\sin\theta - \mu mg\cos\theta = m\ddot{x}$$
Solving this gives $\ddot{x} = g(\sin\theta - \mu\cos\theta)$, and solving this differential equation gives
$$x(t) = x_0 + v_{x0}t + \frac{1}{2}g(\sin\theta - \mu\cos\theta)t^2$$
But since the initial displacement and velocity is 0 in this frame, we have
$$x(t) = \frac{1}{2}g(\sin\theta - \mu\cos\theta)t^2$$

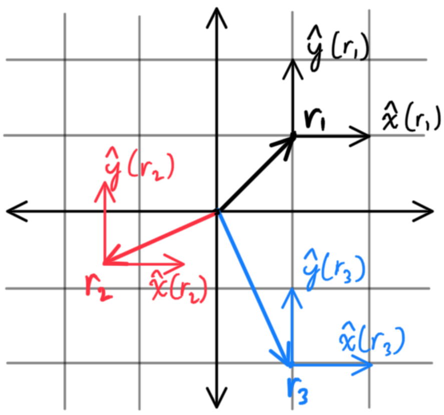
Therefore, the path equation
$$t \mapsto \begin{pmatrix} \frac{1}{2}g(\sin\theta - \mu\cos\theta)t^2 \\ 0 \\ 0 \end{pmatrix}$$
completely determines the motion of the center of mass of the box throughout time.

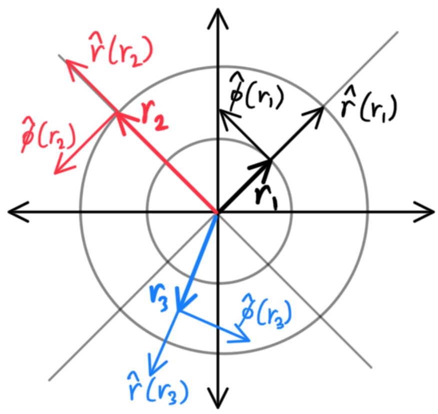
2.1.4 Newton's Laws in 2-Dimensional Polar Coordinates

Recall the polar change of basis transformations
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{x^2+y^2} \\ \arctan(y/x) \end{pmatrix} = \begin{pmatrix} r \\ \phi \end{pmatrix}, \quad \begin{pmatrix} r \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Note that the right transformation is not injective (so the inverse transformation on the left is not even well-defined), so we must restrict their domain and codomains in order to take care of this technical difficulty.

Just as with rectangular coordinates, we introduce two unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$, which are unit vectors that points in the direction of increasing r when ϕ is fixed and increasing ϕ when r is fixed. Unlike unit vectors $\hat{\boldsymbol{x}}, \hat{\mathbf{y}}$ in rectangular coordinates, which point in the same direction no matter where they are located in $\mathbb{R}^2$, the unit vectors $\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}$ point in different directions at different points.





Definition 2.1.13 (Linear Displacement Vector in Polar Coordinates)**.** While the displacement vector of a particle in rectangular coordinates is $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\boldsymbol{y}}$, the displacement vector of a particle in polar coordinates is

$$\mathbf{r} = r\hat{\mathbf{r}} = r\hat{\mathbf{r}} + 0\hat{\boldsymbol{\phi}}$$

Note that this is because the unit vector $\hat{\boldsymbol{r}}$ always points in the direction of the displacement vector.

We now introduce the concept of angular displacement, velocity, and acceleration.

Definition 2.1.14 (Angular Displacement)**.** Given a particle $\boldsymbol{r}(t) = r(t)\,\hat{\boldsymbol{r}}(t)$ at time t, the angular displacement ϕ of it is defined to be be angle it forms with the x-axis.

Definition 2.1.15 (Angular Velocity)**.** The angular velocity is defined as

$$\omega \equiv \dot{\phi}$$

The linear velocity and the angular velocity are not related.

Example 2.1.4 (Fast Angular Velocity, Slow Linear Velocity)**.**

Example 2.1.5 (Slow Angular Velocity, Fast Linear Velocity)**.**

Definition 2.1.16 (Angular Acceleration)**.** The angular acceleration is defined to be

$$\alpha \equiv \dot{\omega} \equiv \ddot{\phi}$$

Discussion on Coordinate System with Changing Unit Vectors

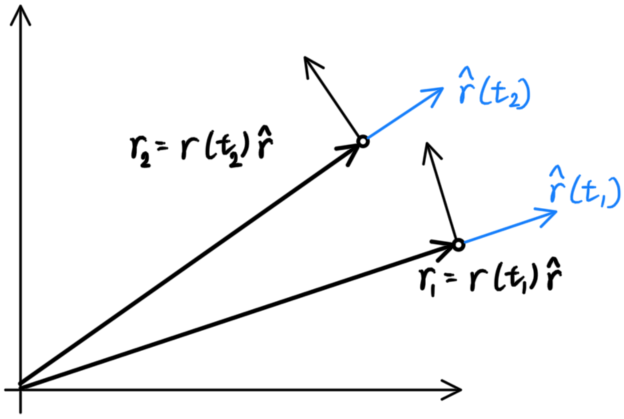
Given the displacement path function $\boldsymbol{r}(t)$ of a particle in motion, so far, we have represented it in terms of the rectangular coordinate system

$$\boldsymbol{r}(t) = x(t)\,\hat{\boldsymbol{x}} + y(t)\,\hat{\boldsymbol{y}}$$

This model is greatly simplified and intuitive, and since the unit vectors themselves aren't changing, we put them in familiar notation$\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}$. However, when in polar form, the notation

$$\boldsymbol{r}(t) = r(t)\hat{\boldsymbol{r}}$$

can be a bit misleading, since the reader also can assume that $\hat{\boldsymbol{r}}$ is also unchanging. However, this is not the case. Say that the particle is at $\boldsymbol{r_1} = r(t_1)\hat{\boldsymbol{r}}$ at time t_1 and $\boldsymbol{r_2} = r(t_2)\hat{\boldsymbol{r}}$ at time t_2, as shown below.



But clearly, the unit vector $\hat{\boldsymbol{r}}$ at t_1 and $\hat{\boldsymbol{r}}$ at t_2 are different, so they themselves are functions of time too! Therefore, the better notation is

$$\boldsymbol{r}(t) = r(t)\,\hat{\boldsymbol{r}}(t)$$

which shows both the unit vector and radius changing with respect to time. Therefore, we have thrown away the assumption that the unit vectors are constant, and rather accept that this unchanging nature of the unit vectors are specific only to the case of the Cartesian coordinate system. So, to be more accurate, we should say that in rectangular coordinates, the displacement vector is

$$\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{x}}(t) + y(t)\hat{\boldsymbol{y}}(t)$$

where $\hat{\boldsymbol{x}}(t), \hat{\boldsymbol{y}}(t)$ are constant and can therefore be represented with the constant vector $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}$.

In order to find the velocity vector, we simply derive the displacement vector with respect to t. By the product rule we get

$$\dot{\boldsymbol{r}}(t) = \dot{r}(t)\hat{\boldsymbol{r}}(t) + r(t)\dot{\hat{\boldsymbol{r}}}(t)$$

and we just have to derive what $\dot{\hat{\boldsymbol{r}}}(t)$ is and put it in terms of the basis $\hat{\boldsymbol{r}}(t), \hat{\boldsymbol{\phi}}(t)$. To do this we require a bit of analysis.

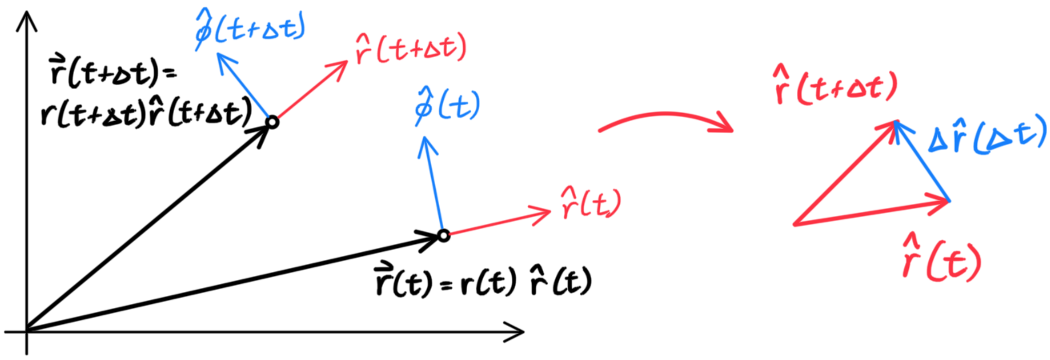
Lemma 2.1.6 (Derivation of Unit Velocity Vectors in 2-Dimensional Polar Coordinates)*.* Given a fixed frame in $\mathbb{R}^2$ in polar coordinates, let the displacement vector of a particle in motion be $\mathbf{r} = r\hat{\mathbf{r}}$. Then, the rate of change of the radial and angular unit vectors are

$$\dot{\hat{\boldsymbol{r}}} = \dot{\phi}\,\hat{\boldsymbol{\phi}}$$
$$\dot{\hat{\boldsymbol{\phi}}} = -\dot{\phi}\,\hat{\boldsymbol{r}}$$

Proof. Given the two vectors at time t and $t + \Delta t$

$$\boldsymbol{r}(t) = r(t)\,\hat{\boldsymbol{r}}(t) \text{ and } \boldsymbol{r}(t+\Delta t) = r(t+\Delta t)\,\hat{\boldsymbol{r}}(t+\Delta t)$$

we focus on their respective unit vectors.



$\Delta\hat{\boldsymbol{r}}(\Delta t) = \hat{\boldsymbol{r}}(t+\Delta t) - \hat{\boldsymbol{r}}(t)$ can be approximated as

$$\Delta\hat{\boldsymbol{r}}(\Delta t) = \Delta\phi(\Delta t)\hat{\boldsymbol{\phi}}(t) + o(1) \text{ as } \Delta t \to 0$$

We can also linearly approximate $\Delta\phi(\Delta t) \equiv \phi(t+\Delta t) - \phi(t)$ (assuming differentiability for simplicity) using its differential to get

$$\begin{aligned}\Delta\phi(\Delta t) \equiv \phi(t+\Delta t) - \phi(t) &= d\phi(t)\,(\Delta t) + o(\Delta t)\\ &= \dot{\phi}(t)\Delta t + o(\Delta t) \text{ as } \Delta t \to 0\end{aligned}$$

Substituting this in the first equation gives

$$\begin{aligned}\Delta\hat{\boldsymbol{r}}(\Delta t) &= \big(\dot{\phi}(t)\Delta t + o(\Delta t)\big)\hat{\boldsymbol{\phi}}(t) + o(1)\\ &= \dot{\phi}(t)\Delta t\hat{\boldsymbol{\phi}}(t) + o(1) \text{ as } \Delta t \to 0\end{aligned}$$

So, taking the derivative gives us

$$\frac{d}{dt}\hat{\boldsymbol{r}}(t) \equiv \lim_{\Delta t\to 0}\frac{\hat{\boldsymbol{r}}(t+\Delta t) - \hat{\boldsymbol{r}}(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta\hat{\boldsymbol{r}}(\Delta t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{\dot{\phi}(t)\Delta t\hat{\boldsymbol{\phi}}(t) + o(1)}{\Delta t} = \dot{\phi}(t)\hat{\boldsymbol{\phi}}(t)$$

■

Definition 2.1.17 (Linear Velocity Vector in Polar Coordinates)**.** Given an inertial frame, the velocity vector for a particle in polar coordinates is

$$\begin{aligned}\boldsymbol{v}(t) &\equiv \dot{r}(t)\,\hat{\boldsymbol{r}}(t) + r(t)\dot{\phi}(t)\,\hat{\boldsymbol{\phi}}(t)\\ &\equiv \dot{r}(t)\,\hat{\boldsymbol{r}}(t) + (r\cdot\dot{\phi})(t)\,\hat{\boldsymbol{\phi}}(t)\end{aligned}$$

or with cleaner notation without the t argument,

$$\begin{aligned}\boldsymbol{v} &\equiv \dot{r}\,\hat{\boldsymbol{r}} + r\dot{\phi}\,\hat{\boldsymbol{\phi}}\\ &\equiv v_r\,\hat{\boldsymbol{r}} + r\omega\,\hat{\boldsymbol{\phi}}\end{aligned}$$

Qualitatively, this means that

1. the r-component of the velocity is equal to the rate at which the radius changes as the particle moves ($\dot{r}$).

2. the ϕ-component of the velocity (the linear rate at which it is spinning around) is equal to the angular velocity of the particle as it moves, scaled by its radius from the origin.

We explain the intuition behind this formula more here. Let us first sketch an arbitrary path of a particle in $\mathbb{R}^2$ in polar coordinates. At arbitrary times t_0, t_1, t_2, the particle will be at points $\boldsymbol{r}(t_0), \boldsymbol{r}(t_1), \boldsymbol{r}(t_2)$, respectively. The velocity vectors at those points are

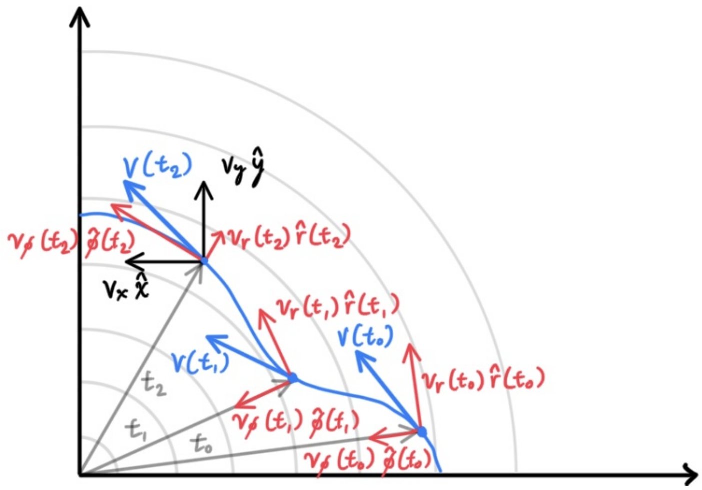
$$\boldsymbol{v}(t_0), \boldsymbol{v}(t_1), \boldsymbol{v}(t_2)$$

which can be represented in terms of two bases: the rectangular basis (represented in black)

$$\boldsymbol{v}(t_i) = v_x(t_i)\,\hat{\boldsymbol{x}} + v_y(t_i)\,\hat{\boldsymbol{y}}$$

and the polar basis (represented in red)

$$\boldsymbol{v}(t_i) = v_r(t_i)\,\hat{\boldsymbol{r}}(t_i) + v_\phi(t_i)\,\hat{\boldsymbol{\phi}}(t_i)$$



Note the procedure here. Given a particle in motion in two-dimensional space, we can visualize this motion by imagining the image of the path function $t \mapsto \boldsymbol{r}(t)$. This also allows us to clearly see the vector $\boldsymbol{v}(t)$ protruding from the point on the path that represents its instantaneous velocity.

Now, upon endowing this space with the polar basis structure, we can represent this abstract velocity vector at point t as a linear combination of its basis vectors $\hat{\boldsymbol{r}}(t), \hat{\boldsymbol{\phi}}(t)$ at point t.

$$\boldsymbol{v}(t) = v_r(t)\,\hat{\boldsymbol{r}}(t) + v_\phi(t)\,\hat{\boldsymbol{\phi}}(t)$$

This means that at time t, when the particle is at point $\boldsymbol{r}(t) = r(t)\,\hat{\boldsymbol{r}}(t)$, the basis vectors at that instant of time points outwards from the radius ($\hat{\boldsymbol{r}}(t)$) and leftwards perpendicular to the first vector ($\hat{\boldsymbol{\phi}}(t)$). Then,

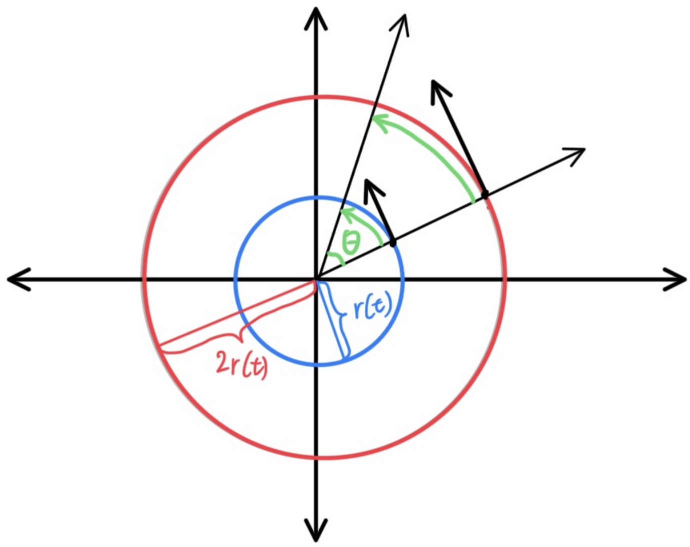
1. The r-component of $\boldsymbol{v}$, $v_r(t)\,\hat{\boldsymbol{r}}(t)$, represents the instantaneous rate at which the particle is moving away from the origin; that is, the rate at which the radius is increasing.

2. The ϕ-component of $\boldsymbol{v}$, $v_\phi(t)\,\hat{\boldsymbol{\phi}}(t)$, represents the instantaneous rate at which the particle is moving around the origin; that is, the rate at which the angle θ is increasing.

To interpret the formula, remember that the component coefficients v_x and v_y are scalars, and so they have no say in the direction of the velocity vectors (direction is determined by the unit vectors). Therefore, when getting a sense of these component functions, all we have to do is compare them to the magnitude of the component vectors $v_r\,\hat{\boldsymbol{r}}, v_\phi\,\hat{\boldsymbol{\phi}}$.

To see why the r-component of $\boldsymbol{v}$ is $\dot{r}$, it makes visual sense that a particle spiraling outwards from the origin, meaning that it has a velocity vector that also points outwards, is increasing in radius r. Same for spiraling inwards. Therefore, the r component of the velocity is simply the rate of change of the radius itself.

The ϕ-component is slightly harder to interpret. Remember that the ϕ component is just the angular velocity: the rate at which the particle spins around the origin (i.e. angle changing). Lets assume that the particle circles the origin at a constant radius $r(t)$ for all t. Its velocity vector is clearly moving in just the ϕ-direction (and its r-component is 0). If the same particle were to circle the origin at double the original radius $2r(t)$ for all t, then since it must travel twice as much (since the circle's circumference is doubled) it must travel at twice the speed to have the same angular velocity.



This line of reasoning leads to the intuition that given a particle's path of motion $\boldsymbol{r}$, the further the r-component of $r\hat{\boldsymbol{r}}$ (i.e. the further it is away from the origin), the more linear velocity v_ϕ in the direction $\hat{\boldsymbol{\phi}}$ is must have in order to have the same angular velocity. Therefore, we must take the angular velocity $\dot{\phi}$ and scale it by the radius length r to get the linear velocity $\dot{\phi}r$ of the particle in direction of unit vector $\hat{\boldsymbol{\phi}}$ at time t.

Definition 2.1.18 (Linear Acceleration Vector in Polar Coordinates)**.** The acceleration vector for a particle in polar coordinates is quite complicated. It is

$$\boldsymbol{a}(t) \equiv \big(\ddot{r}(t) - r(t)\dot{\phi}^2(t)\big)\hat{\boldsymbol{r}}(t) + \big(r(t)\ddot{\phi}(t) + 2\dot{r}(t)\dot{\phi}(t)\big)\hat{\boldsymbol{\phi}}(t)$$

or in cleaner notation

$$\boldsymbol{a} \equiv \big(\ddot{r} - r\dot{\phi}^2\big)\hat{\boldsymbol{r}} + \big(r\ddot{\phi} + 2\dot{r}\dot{\phi}\big)\hat{\boldsymbol{\phi}}$$

Explaining this qualitatively too complicated, so we present a simpler form for when the radius r is constant, say $r(t) = R$ for all t.

$$\begin{aligned}\boldsymbol{a} &\equiv -R\dot{\phi}^2\,\hat{\boldsymbol{r}} + R\ddot{\phi}\,\hat{\boldsymbol{\phi}} \\ &\equiv -R\omega^2\,\hat{\boldsymbol{r}} + R\alpha\,\hat{\boldsymbol{\phi}}\end{aligned}$$

This means that for a particle that moves around a fixed circle,

1. the r-component of the acceleration vector is $-R\omega^2$, meaning that the particle is accelerating inwards at a rate of

$$R\omega^2 = \frac{v^2}{R}$$

known as the *centripetal acceleration.*

2. the ϕ-component of the acceleration vector is $R\alpha$, representing the value of the tangential acceleration.

Theorem 2.1.7 (Newton's Second Law, Polar Form). Having calculated linear acceleration in polar coordinates, we can finally write down Newton's second law in terms of polar coordinates.

$$\boldsymbol{F} = m\boldsymbol{a} \iff \begin{pmatrix} F_r \\ F_\phi \end{pmatrix} = m \begin{pmatrix} \ddot{r} - r\dot{\phi}^2 \\ r\ddot{\phi} + 2\dot{r}\dot{\phi} \end{pmatrix}$$

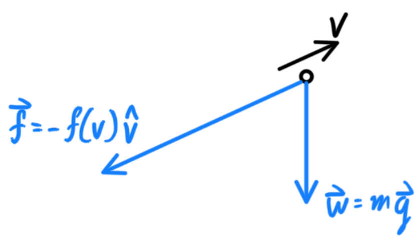
Example 2.1.6 (Oscillating Skateboard)**.**

2.2 Projectiles and Charged Particles

Definition 2.2.1 (Air Resistance)**.** The resistive force, or *drag*, $\boldsymbol{f}$ of air or any other medium has some basic properties that we should be familiar with:

1. It depends (and is often proportional) to the speed v of the object concerned.
2. The direction of the force due to motion through the air is usually opposite to the velocity $\boldsymbol{v}$ (even though there are some exceptional cases where it isn't opposite). We will assume this from now.

Therefore, given a particle traveling through a medium with a certain velocity on Earth, the two forces acting on it are the gravitational force $\boldsymbol{w} = m\boldsymbol{g}$ and the drag force of air resistance $\boldsymbol{f} = -f(v)\hat{\boldsymbol{v}}$.



The function f can be complicated, especially as the object's speed approaches the speed of sound. However, at lower speeds it is often a good approximation to write

$$f(v) = bv + cv^2 = f_{lin} + f_{quad}$$

However, sometimes one of the linear/quadratic terms may be small compared to the other and can therefore be negligible.

1. Very small liquid drops in air and slightly larger objects in a very viscous fluid have drag forces that are dominantly linear, and so
$$\boldsymbol{f} = -bv\hat{\boldsymbol{v}} = -b\boldsymbol{v}$$

2. Most projectiles, such as golf balls, cannonballs, and humans in free fall, have drag forces that are dominantly quadratic, meaning
$$\boldsymbol{f} = -cv^2\hat{\boldsymbol{v}} = -cv\boldsymbol{v}$$

Definition 2.2.2 (Air Resistance of Spherical Projectile at STP)**.** In standard temperature and pressure (STP), the air resistance relation $f(v)$ of a spherical projectile (cannonball, baseball, drop of rain) is
$$f(v) = \beta D v + \gamma D^2 v^2$$
where

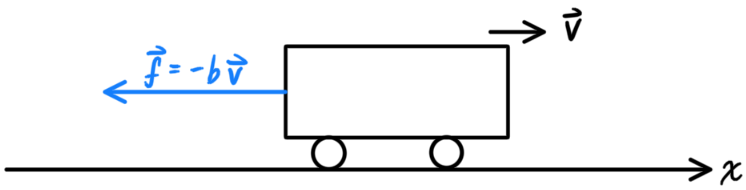
1. D denotes the diameter of the projectile
2. $\beta = 1.6 \cdot 10^{-4}\ N \cdot s/m^2$
3. $\gamma = 0.25\ N \cdot s^2/m^4$

Example 2.2.1 (Equation of Motion of a Particle)**.** *Given the particle with initial velocity* $\boldsymbol{v_0}$ *with drag force* $\boldsymbol{f} = -f(v)\hat{\boldsymbol{v}}$ *and gravitational force* $\boldsymbol{w} = m\boldsymbol{g}$*, we see that the total force* $\boldsymbol{F}$ *is the sum of these two forces. This leads to the differential equation*
$$m\ddot{\boldsymbol{r}} = m\boldsymbol{g} + \boldsymbol{f} = m\boldsymbol{g} - f(v)\hat{\boldsymbol{v}} \implies m\ddot{\boldsymbol{r}}(t) = m\boldsymbol{g} - f\big(v(t)\big)\hat{\boldsymbol{v}}(t)$$
where we can solve it by interpreting $\boldsymbol{v} = \dot{\boldsymbol{r}}$*. If* $\boldsymbol{f} = -b\boldsymbol{v}$*, then*
$$m\ddot{\boldsymbol{r}}(t) = m\boldsymbol{g} - b\boldsymbol{v}(t) \implies m\ddot{\boldsymbol{r}}(t) = m\boldsymbol{g} - b\dot{\boldsymbol{r}}(t)$$
which can be solved as a first-order differential equation in $\boldsymbol{v}$*.*
$$m\dot{\boldsymbol{v}}(t) = m\boldsymbol{g} - b\boldsymbol{v}(t)$$
with initial value $\boldsymbol{v}(0) = \boldsymbol{v_0}$*.*

2.2.1 Trajectory and Range in a Linear Medium

Horizontal Motion with Linear Drag

Consider an object (such as a cart) coasting horizontally in a linearly resistive medium, with initial displacement $x = 0, v_x = v_{x0}$. With the proper frame, we can just focus on the x-component of motion.



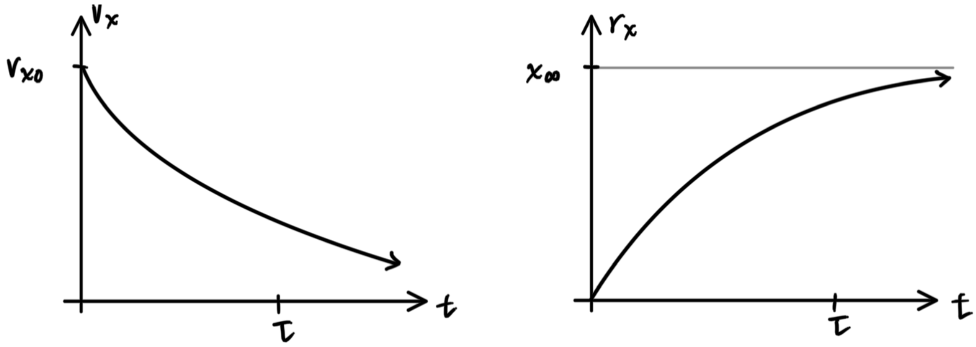
The only force on the cart is the drag $\boldsymbol{f}(t) = -b\boldsymbol{v}(t)$. Therefore, the equation of motion in the x-component is

$$m\dot{\boldsymbol{v}}_{\boldsymbol{x}}(t) = -b\boldsymbol{v}_{\boldsymbol{x}}(t) \implies \dot{\boldsymbol{v}}_{\boldsymbol{x}}(t) = -\frac{b}{m}\boldsymbol{v}_{\boldsymbol{x}}(t)$$

which has the solution

$$v_x(t) = v_{x0}e^{-t/\tau}, \quad \tau = \frac{m}{b} \text{ represents linear drag}$$

We can see that the cart slows down exponentially, which makes sense as shown in the graph.



where $x_\infty = v_{x0}\tau$ is the point where the cart will converge to.

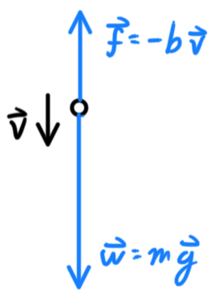
Lemma 2.2.1 (Horizontal Motion of Particle with Linear Drag). Integrating the solution for v_x gives the equation of horizontal motion of a particle with linear drag as

$$x(t) = x_0 + v_{x0}\tau\left(1 - e^{-t/\tau}\right)$$

where x_0 is the initial displacement and v_{x0} is the initial velocity.

Vertical Motion with Linear Drag

Let us consider a projectile that is subject to linear air resistance and is thrown vertically downward.



The two forces on the projectile are gravity and air resistance. Focusing on the y-component (with the appropriate frame where $y > 0$ is downwards), we have

$$m\dot{v}_y = mg - bv_y$$

With the velocity downward, the retarding force is upward, while the force of gravity is downward.

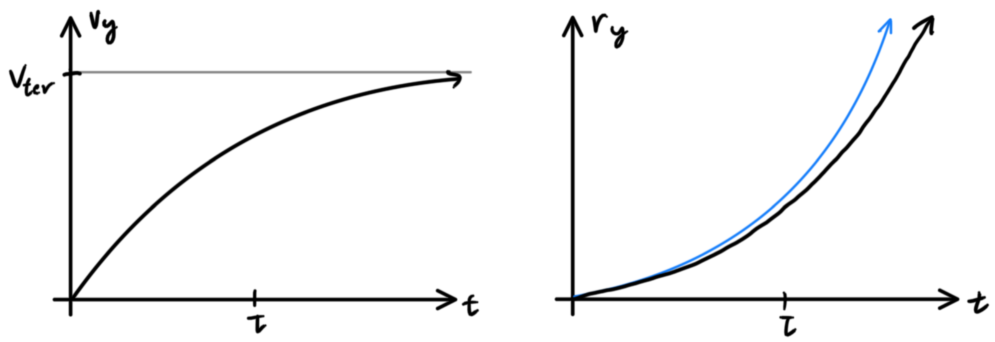
Definition 2.2.3 (Terminal Velocity)**.** If v_y is small, the force of gravity overpowers the drag force and so the particle accelerates downwards. This will continue until the drag force balances the weight. This speed is easily found by setting the acceleration to 0, leading to the *terminal speed* formula

$$v_{ter} = \frac{mg}{b}$$

Clearly this is only the y-component of velocity, and the actual vector form would be

$$\boldsymbol{v_{ter}} = \begin{pmatrix} 0 \\ mg/b \end{pmatrix}$$

Just like horizontal motion, when an object is dropped in a medium with linear resistance, v_y approaches its terminal value v_{ter} exponentially in the following manner.



Lemma 2.2.2 (Vertical Motion of Particle with Linear Drag). Integrating the solution for v_x gives the equation of vertical motion of a particle (in a gravitational field) with linear drag (where $y > 0$ points downward) as

$$y(t) = y_0 + v_{ter}t + (v_{y0} - v_{ter})\tau\left(1 - e^{-t/\tau}\right)$$

or, when $y > 0$ points upward,

$$y(t) = y_0 + (v_{y0} + v_{ter})\tau\left(1 - e^{-t/\tau}\right) - v_{ter}t$$

where y_0 is the initial displacement and v_{y0} is the initial velocity.

General Trajectory with Linear Drag

We can combine the horizontal and vertical components into a general solution in $\mathbb{R}^2$.

Theorem 2.2.3 (Motion of Particle in Gravitational Field with Linear Drag). Given a mass-m particle in motion within a gravitational field (with acceleration $-g$) in $\mathbb{R}^2$ (with frame $x, y > 0$ pointing right, up), let its initial displacement and velocity (at $t = 0$) be

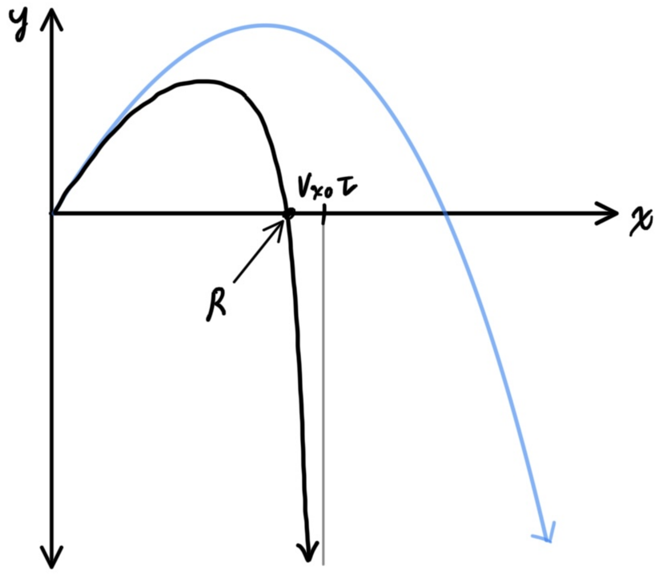
$$\boldsymbol{r}(0) \equiv \boldsymbol{r_0} \equiv \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \text{ and } \boldsymbol{v}(0) \equiv \boldsymbol{v_0} \equiv \begin{pmatrix} v_{x0} \\ v_{y0} \end{pmatrix}$$

Then, the equation of motion of the particle, shown component-wise, is

$$x(t) = x_0 + v_{x0}\tau\big(1 - e^{-t/\tau}\big) \tag{2.1}$$

$$y(t) = y_0 + (v_{y0} + v_{ter})\tau\big(1 - e^{-t/\tau}\big) - v_{ter}t \tag{2.2}$$

where $\tau = m/b$ and the terminal velocity $v_{ter} = -mg/b$. This curve can be plotted and compared with the motion of a particle in vacuum (without drag force). Notice how the ball's trajectory with drag force is asymptotic as $x \to v_{x0}\tau$.



Corollary 2.2.3.1 (Equation for Trajectory). By removing the parameter t, we can turn the paramateric equation into an equation of x and y

$$y = y_0 + \frac{v_{y0} + v_{ter}}{v_{x0}}\big(x - x_0\big) + v_{ter}\tau \ln\left(1 - \frac{x - x_0}{v_{x_0}\tau}\right)$$

or by choosing the appropriate frame where the origin is (x_0, y_0), we have

$$y = \frac{v_{y0} + v_{ter}}{v_{x0}}x + v_{ter}\tau \ln\left(1 - \frac{x}{v_{x_0}\tau}\right)$$

Definition 2.2.4 (Horizontal Range of a Particle)**.** The horizontal range R of a projectile in a vacuum is

$$R_{vac} = \frac{2v_{x0}v_{y0}}{g}$$

However, the horizontal range of it subject to linear air resistance is the nonzero solution R to the equation

$$0 = \frac{v_{y0} + v_{ter}}{v_{x0}}R + v_{ter}\tau \ln\left(1 - \frac{R}{v_{x_0}\tau}\right)$$

which cannot be represented in terms of elementary functions. However, we can use various methods, such as numerical methods (Euler, Simpson, Runge-Kutta) or since it is reasonable to assume that v_{x0}, τ are large, we can assume $\frac{R}{v_{x0}\tau}$ small to use the kth degree Taylor approximation

$$\ln(1 - \epsilon) = -\left(\epsilon + \frac{1}{2}\epsilon^2 + \frac{1}{3}\epsilon^3 + \ldots + \frac{1}{k}\epsilon^k\right)$$

to convert the expression into a polynomial equation in terms of R.

For example, the third-degree approximation of $\ln(1-\epsilon)$ leads to the equation

$$\left(\frac{v_{y0}+v_{ter}}{v_{x0}}\right)R - v_{ter}\tau\left(\frac{R}{v_{x0}\tau}+\frac{1}{2}\left(\frac{R}{v_{x0}\tau}\right)^2+\frac{1}{3}\left(\frac{R}{v_{x0}\tau}\right)^3\right)=0$$

and the solution

$$R=\frac{2v_{x0}v_{y0}}{g}-\frac{2}{3v_{x0}\tau}R^2\approx R_{vac}$$

2.2.2 Trajectory and Range in a Quadratic Medium

Modeling quadratic air resistance also requires us to solve the differential equation

$$m\dot{\boldsymbol{v}}=m\boldsymbol{g}+\boldsymbol{f}$$

but this becomes a *nonlinear* differential equation, which is considerably harder to solve.

Horizontal Motion with Quadratic Drag

Given a cart coasting horizontally in a quadratically resistive medium with initial displacement $x=0, v_x=v_{x0}$. The only force on the cart is the drag $\boldsymbol{f}(t)=-c||\boldsymbol{v}(t)||\boldsymbol{v}(t)$, and so we get the equation

$$m\dot{\boldsymbol{v}}(t)=-cv(t)\boldsymbol{v}(t) \iff m\begin{pmatrix}\dot{v_x}(t)\\0\end{pmatrix}=-c||\boldsymbol{v}(t)||\begin{pmatrix}v_x(t)\\0\end{pmatrix}$$

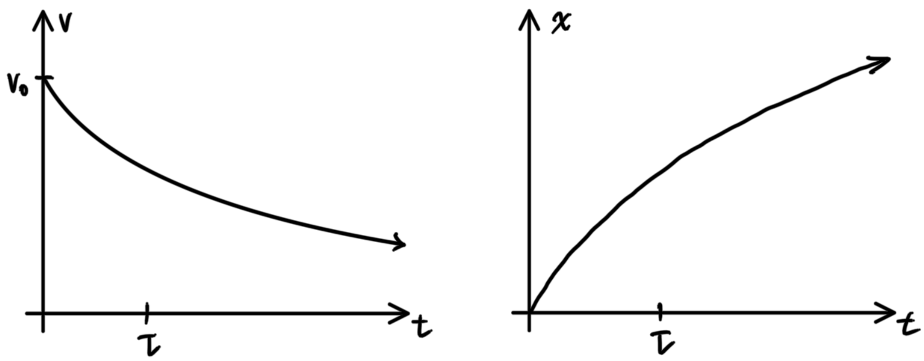
Focusing on the x-component gives

$$m\dot{v_x}(t)=-c\big(v_x(t)\big)^2$$

which is a differential equation that can be solved using variable-separation. Integrating it gives

$$v_x(t)=\frac{v_{x0}}{1+cv_0t/m}=\frac{v_0}{1+t/\tau},\qquad \tau=\frac{m}{cv_0}\text{ represents quadratic drag}$$

We can visualize the motions with the following graphs



Note the similarities and differences between the graphs of a particle in motion with linear drag and with quadratic drag. Specifically,

1. both velocities go to 0 as $t \to \infty$
2. In the linear case, v_x goes to 0 exponentially, while in the quadratic case $v_x \to 0$ slowly, like $1/t$.
3. In the linear case, x approaches a finite limit as $t \to \infty$, while in the quadratic case x increases without limit as $t \to \infty$.

Lemma 2.2.4 (Horizontal Motion of Particle with Quadratic Drag)*.* Integrating the solution for v_x gives the equation of horizontal motion of a particle with quadratic drag as

$$x(t) = x_0 + v_{x0}\tau \ln\left(1 + \frac{t}{\tau}\right)$$

where x_0 is the initial displacement and v_{x0} is the initial velocity.

Vertical Motion with Quadratic Drag

Given a projectile that is subject to quadratic drag force and is thrown vertically downward, the two forces on the projectile are gravity and air resistance. Measuring the y-coordinate vertically down, the equation of motion is

$$m\dot{v_y} = mg - cv_y^2$$

The terminal speed of the projectile can be calculated by setting the left hand side to 0, giving

$$v_{ter} = \sqrt{\frac{mg}{c}}$$

solving for c and substituting into the differential equation gives

$$\dot{v_y} = g\left(1 - \frac{v_y^2}{v_{ter}^2}\right)$$

leading to the solution

$$v_y(t) = v_{ter} \tanh\left(\frac{gt}{v_{ter}} + \operatorname{arctanh}\left(\frac{v_{y0}}{v_{ter}}\right)\right)$$

where v_{y0} is the initial velocity of the particle.

Lemma 2.2.5 (Vertical Motion of Particle with Quadratic Drag)*.* Integrating the solution for v_y gives the equation of vertical motion of a particle with quadratic drag as

$$y(t) = y_0 + \frac{v_{ter}^2}{g} \ln\left(\cosh\left(\frac{gt}{v_{ter}} + \operatorname{arccot}\left(\frac{v_{y0}}{v_{ter}}\right)\right)\right)$$

where y_0 is the initial displacement and v_{y0} is the initial velocity.

General Trajectory with Quadratic Drag

Theorem 2.2.6 (Motion of Particle in Gravitational Field with Quadratic Drag). Given a mass-m particle in motion within a gravitational field (with acceleration $-g$) in $\mathbb{R}^2$ (with frame $x, y > 0$ pointing right, up), let its initial displacement and velocity (at $t = 0$) be

$$\boldsymbol{r}(0) \equiv \boldsymbol{r_0} \equiv \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \text{ and } \boldsymbol{v}(0) \equiv \boldsymbol{v_0} \equiv \begin{pmatrix} v_{x0} \\ v_{y0} \end{pmatrix}$$

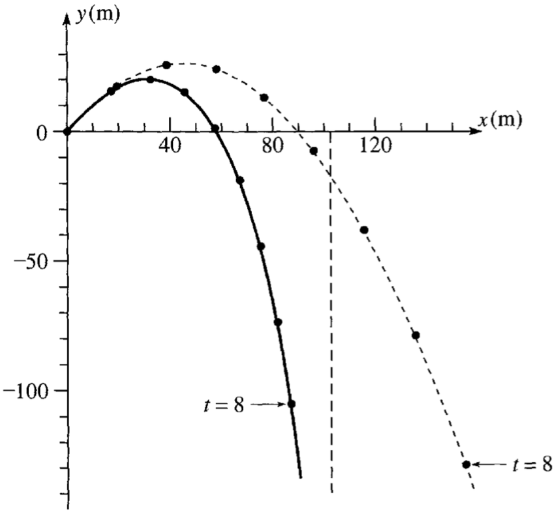
Then, the solutions to the differential equation representing the motion

$$m\ddot{\boldsymbol{r}} = m\boldsymbol{g} - cv\boldsymbol{v} \iff \begin{cases} m\dot{v_x} = -c\sqrt{v_x^2 + v_y^2}\, v_x \\ m\dot{v_y} = -mg - c\sqrt{v_x^2 + v_y^2}\, v_y \end{cases}$$

are not solvable analytically. Unlike the linear terms, the component equations contain variables that are dependent on the other equation, so they cannot be solved independently. Therefore, they must be solved numerically.

However, there are properties that hold for all solutions to the system of equations.

Example 2.2.2 (Trajectory of a Baseball). *The graph shows the trajectory of a baseball thrown off a cliff and subject to quadratic air resistance. The initial velocity is* 30 *m/s at* 50° *above the horizontal. The terminal speed is* 35 *m/s. The dashed curve shows the corresponding trajectory in a vacuum, and the dots show the ball's position at one-second time intervals.*



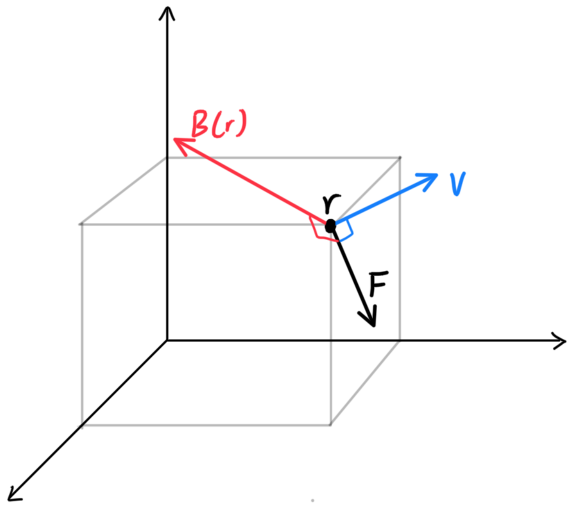
The effect of air resistance is quite large in this example since the ball was thrown only little less than the terminal speed, meaning that the force of air resistance is only a little less than that of gravity.

2.2.3 Motion of a Charge in a Magnetic Field

Lemma 2.2.7 (Motion of Charged Particle in Magnetic Field). Let a particle in $\mathbb{R}^3$ have charge q, moving in velocity $\boldsymbol{v}$ at an instant in time. Let us have a magnetic field $\boldsymbol{B} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$. Then, the net force on the particle at that instant is

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$$

where $\times$ represents the cross product in $\mathbb{R}^3$.



We can visualize the direction of this force using the familiar *right-hand rule*.

Solving for the motion of charged particles in a magnetic field is similar to that of projectiles in a gravitational field. In both cases, we get an initial velocity (initial push of charged particle vs. initial throw of projectile). We find all the forces that are acting on this particle, changing the velocity over time, and by solving the differential equation, we find the equation for $\boldsymbol{v}$. Integrating $\boldsymbol{v}(t)$ over t and using the fundamental theorem of calculus gives the equation of motion $\boldsymbol{r}$.

Motion of Charge in Uniform Magnetic Field

Let us consider a particle of mass m and charge q moving within a uniform magnetic field $\boldsymbol{B}$ at initial velocity and displacement

$$\boldsymbol{v_0} = \begin{pmatrix} v_{x0} \\ v_{y0} \\ v_{z0} \end{pmatrix} \text{ and } \boldsymbol{r_0} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

By choosing the appropriate frame, I can assume $\boldsymbol{B}$ to point only in the z-direction. Therefore, the net force on the particle is just the magnetic force, leading to the differential equation

$$m\dot{\boldsymbol{v}}(t) = q\boldsymbol{v}(t) \times \boldsymbol{B} \iff m \begin{pmatrix} \dot{v_x} \\ \dot{v_y} \\ \dot{v_z} \end{pmatrix} = q \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}$$

leads to the system of equations

$$\begin{aligned} m\dot{v_x} &= qBv_y \\ m\dot{v_y} &= -qBv_x \\ m\dot{v_z} &= 0 \end{aligned}$$

The third equation obviously assumes that v_z is constant, and so the solution for the z-component is $z(t) = z_0 + v_{z0}t$. As for the first two coupled equations, we introduce the constant $\omega = \frac{qB}{m}$ called the *cyclotron frequency* to simplify them into

$$\dot{v_x} = \omega v_y$$
$$\dot{v_y} = -\omega v_x$$

which has solution

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} \cos(-\omega t) & -\sin(-\omega t) \\ \sin(-\omega t) & \cos(-\omega t) \end{pmatrix} \begin{pmatrix} v_{x0} \\ v_{y0} \end{pmatrix} = \begin{pmatrix} v_{x0}\cos(-\omega t) & -v_{y0}\sin(-\omega t) \\ v_{x0}\sin(-\omega t) & v_{y0}\cos(-\omega t) \end{pmatrix}$$

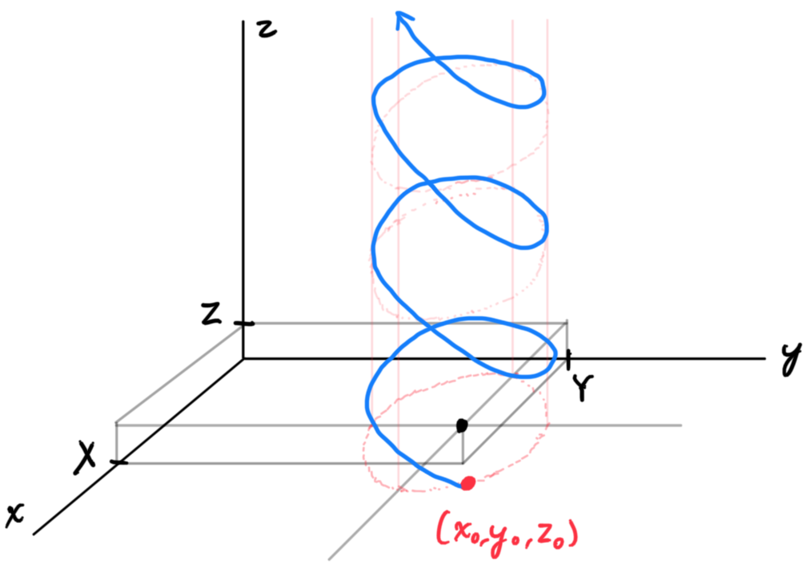
Therefore, the solutions to the system are

$$\begin{aligned} x(t) &= X - \frac{1}{\omega}v_{x0}\sin(-\omega t) - \frac{1}{\omega}v_{y0}\cos(-\omega t) \\ y(t) &= Y + \frac{1}{\omega}v_{x0}\cos(-\omega t) - \frac{1}{\omega}v_{y0}\sin(-\omega t) \\ z(t) &= z_0 + v_{z0}t \end{aligned}$$

where X, Y (the center of the helicoid) are determined so that $x(0) = X, y(0) = Y$. We can put this in the form of a rotation matrix to get the solution

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} X \\ Y \\ z_0 \end{pmatrix} + \frac{1}{\omega}\begin{pmatrix} \sin(\omega t) & -\cos(\omega t) & 0 \\ \cos(\omega t) & \sin(\omega t) & 0 \\ 0 & 0 & t \end{pmatrix} \begin{pmatrix} v_{x0} \\ v_{y0} \\ v_{z0} \end{pmatrix}$$

which traces out a helicoid rotating clockwise.



Note that the radius of the orbit is

$$r = \frac{v}{\omega} = \frac{mv}{qB} = \frac{p}{qB}$$

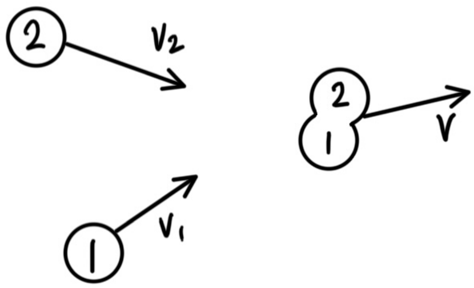
The radius increases as the particles accelerate, so that they eventually emerge at the outer edge of the circular magnets that produce the magnetic field.

2.3 Momentum, Angular Momentum

Recall the principle of conservation of momentum.

Theorem 2.3.1. If the net external force $\boldsymbol{F^{ext}}$ on an N-particle system is 0, the system's total mechanical momentum $\boldsymbol{P} = \sum m_i v_i$ is constant.

Example 2.3.1 (Perfectly Inelastic Collision of Two Bodies)**.** *Two bodies have masses m_1, m_2 and velocities $\boldsymbol{v_1}, \boldsymbol{v_2}$. The two bodies collide and lock together, moving off as a single unit. Assuming no external forces in the system, find the velocity $\boldsymbol{v}$ of the single body just after the collision.*



Since there are no external forces acting on the system, by the law of conversation of momentum the total momentum of the two bodies and that of the single body must be the same. Therefore,

$$m_1\boldsymbol{v_1} + m_2\boldsymbol{v_2} = (m_1 + m_2)\boldsymbol{v} \implies \boldsymbol{v} = \frac{m_1\boldsymbol{v_1} + m_2\boldsymbol{v_2}}{m_1 + m_2}$$

A special case if when one of the bodies is initially at rest. With $\boldsymbol{v_2} = 0$, we have

$$\boldsymbol{v} = \frac{m_1}{m_1 + m_2}\boldsymbol{v_1}$$

Rocket Science

In order for a rocket to get off of the ground, it must spurt out a bunch of fuel towards the ground, allowing it to lift off by the law of conservation of momentum. Since we can simplify the upwards motion of a rocket into one-dimension, let us focus on its y-component.

Consider a rocket with initial mass m_0, traveling upwards in the positive y-direction and ejecting fuel at the constant exhaust speed v_{ex} relative to the rocket. Since the rocket is ejecting mass, the rocket's mass $m(t_0)$ is steadily decreasing and can be modeled as a function of time

$$m(t) \equiv m_0 - rt$$

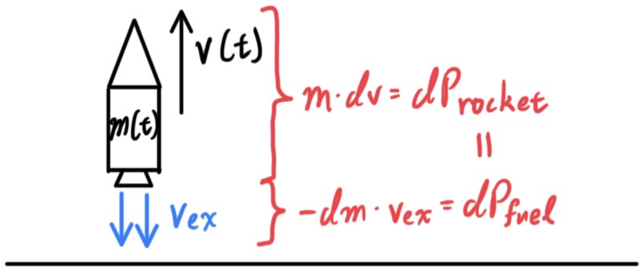
This function must be known beforehand. Let the speed of the rocket be represented with $v(t)$, which must be figured out. We can define a function of the momentum

$$P_{rocket}(t) = m(t)v(t)$$

and at time $t+\Delta t$, we have

$$\begin{aligned} P_{rocket}(t+\Delta t) &= m(t+\Delta t)\,v(t+\Delta t) \\ &= \big(m(t)+\Delta m(\Delta t)\big)\big(v(t)+\Delta v(\Delta t)\big) \end{aligned}$$

where clearly $\Delta m(\Delta t) < 0$ if $\Delta t > 0$ since fuel is spurting out from the rocket. This means that the fuel ejected in the time Δt has mass $-\Delta m(\Delta t)$ and velocity $v(t) - v_{ex}$ relative to the ground.



For small Δt, we can approximate this using the differentials

$$P_{rocket}(t+\Delta t) = \big(m(t)+dm(\Delta t)\big)\big(v(t)+dv(\Delta t)\big) + o(\Delta t)$$

and so the total momentum of the rocket plus the fuel ejected at $t+\Delta t$ is

$$\begin{aligned} P(t+\Delta t) &= \big(m(t)+dm(\Delta t)\big)\big(v(t)+dv(\Delta t)\big) - dm(\Delta t)\big(v(t)-v_{ex}\big) + o(\Delta t) \\ &= m(t)v(t) + m(t)dv(\Delta t) + v_{ex}dm(\Delta t) + o(\Delta t) \\ &= P(t) + dP(\Delta t) + o(\Delta t) \end{aligned}$$

and so the differential dP is

$$dP(\Delta t) = m(t)dv(\Delta t) + v_{ex}dm(\Delta t)$$

If there is a net external force F^{ext} such as gravity, this change of momentum is $dP(\Delta t) = F^{ext}dt(\Delta t)$. But assuming no external forces implies $dP = 0$ and so

$$m(t)dv(\Delta t) = -v_{ex}dm(\Delta t)$$

dividing both sides by $dt(\Delta t)$ gives

$$m(t)\frac{dv(t)(\Delta t)}{dt(t)(\Delta t)} = -v_{ex}\frac{dm(t)(\Delta t)}{dt(t)(\Delta t)}$$

or without the argument Δt, we have

$$m(t)\frac{d}{dt}(t) = -v_{ex}\frac{d}{dt}m(t) \iff m(t)\dot{v}(t) = -v_{ex}\dot{m}(t)$$

where $-\dot{m}$ is the rate at which the rocket engine is ejecting mass.

Definition 2.3.1 (Thrust)**.** The *thrust* of a rocket of changing mass $m(t)$ and with constant exhaust speed v_{ex} is defined

$$\text{Thrust} = -\dot{m}v_{ex}$$

and since $\dot{m}$ is negative, this defines thrust to be positive.

Lemma 2.3.2 (Equation of Motion of a Rocket). Given a rocket with initial mass m_0, initial velocity v_0, and mass function $m(t)$ that represents the mass of the rocket over time, the differential equation of motion of a rocket with no external forces

$$m\,dv = -v_{ex}\,dm$$

can be solved to give

$$v = v_0 + v_{ex} \ln\left(\frac{m_0}{m}\right)$$

This result puts a significant restriction on the maximum speed of the rocket. The ratio m_0/m is largest when all the fuel is burned and m is just the mass of rocket plus payload. So even if the original mass is 90% fuel, this ratio is 10 meaning that $\ln 10 = 2.3$. Therefore, the speed gained cannot be more than $2.3v_{ex}$.

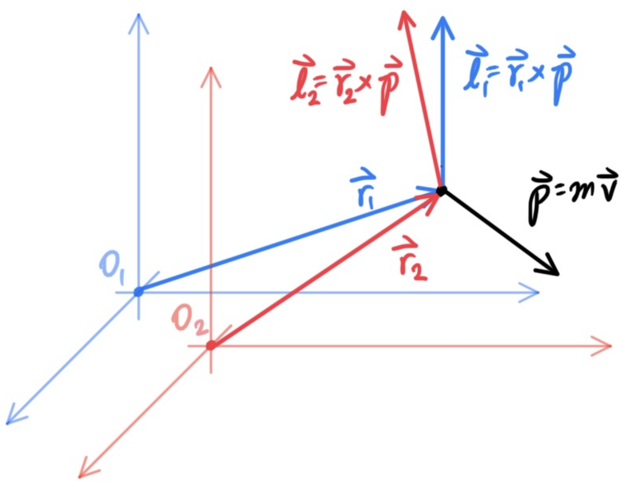
2.3.1 Angular Momentum of a Single Particle

We have went over the center of mass and properties of the linear momentum of a system of particles. We now move onto angular momentum.

Definition 2.3.2 (Angular Momentum)**.** Within $\mathbb{R}^3$ with a frame, the *angular momentum* $\boldsymbol{\ell}$ of a single particle is defined as the vector

$$\boldsymbol{\ell} \equiv \boldsymbol{r} \times \boldsymbol{p}$$

Note that unlike linear momentum $\boldsymbol{p}$, since $\boldsymbol{r}$ is dependent on the choice of origin O, $\boldsymbol{\ell}$ is also dependent on the choice of O. Therefore, we refer to $\boldsymbol{\ell}$ as the angular momentum relative to O.



Definition 2.3.3 (Torque, Rotational Form of Newton's Second Law)**.** The *torque* of a particle about origin O is defined as the derivative of its angular momentum with respect to time. That is,

$$\boldsymbol{\Gamma} \equiv \dot{\boldsymbol{\ell}}$$

That is, $\mathbf{\Gamma}$ represents how fast the angular momentum changes. With a bit of calculations, we can see that

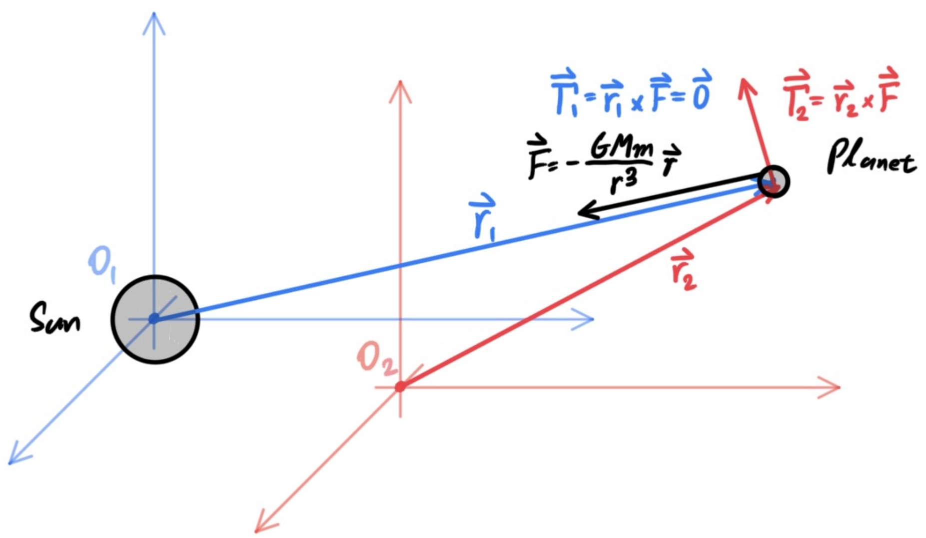
$$\begin{aligned}\mathbf{\Gamma} &= \frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{p}) = (\dot{\boldsymbol{r}} \times \boldsymbol{p}) + (\boldsymbol{r} \times \dot{\boldsymbol{p}}) \\ &= \boldsymbol{r} \times \boldsymbol{F}\end{aligned}$$

where $\boldsymbol{F}$ is the linear momentum of the particle.

Kepler's Second Law

In many one particle problems one can choose the origin O so that the net torque $\mathbf{\Gamma}$ (about the chosen O) is zero, which implies that the angular momentum about O is constant.

Example 2.3.2 (Orbit Around the Sun)**.** *Let a single planet orbit the sun. The only force on the planet is the gravitational pull GmM/r^2 of the sun.*



Therefore, $\boldsymbol{F}$ is antiparallel to $\boldsymbol{r}$ and therefore the torque of the planet vanishes.

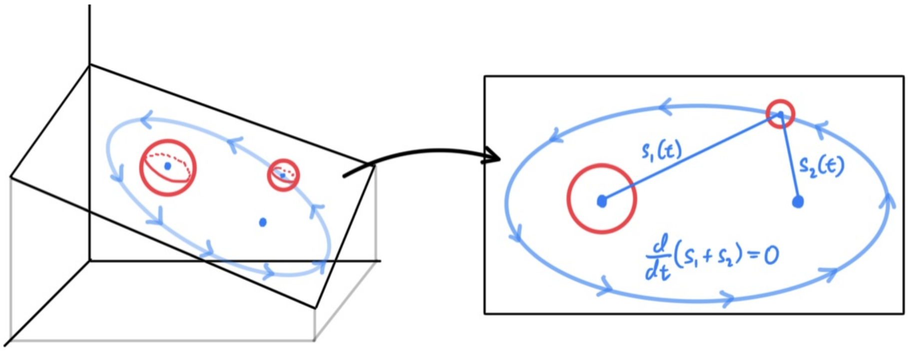
$$\mathbf{\Gamma} = \boldsymbol{r} \times \boldsymbol{F} = 0$$

Thus, choosing our origin at the sun implies that the planet's angular momentum about O is constant, allowing us to greatly simplify our problem. For example, because $\boldsymbol{r} \times \boldsymbol{p}$ is constant, $\boldsymbol{r}, \boldsymbol{p}$ must remain in a fixed plane and therefore the planet's orbit is confined to a single two-dimensional plane.

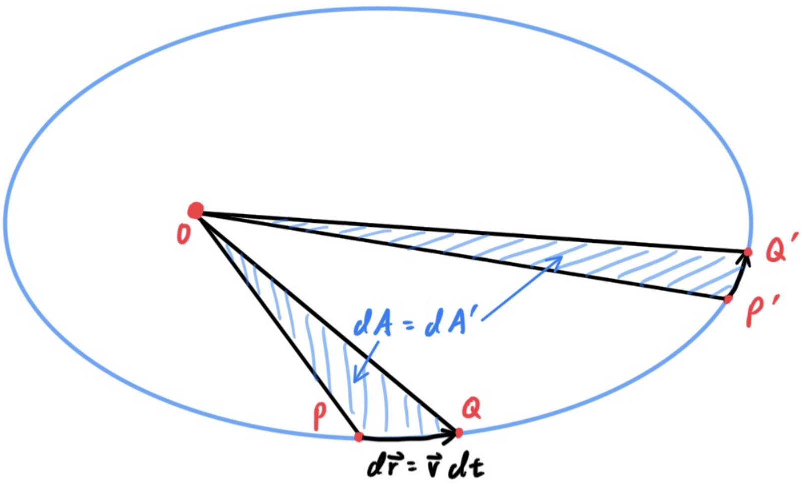
With this, we can now derive Kepler's second law. Unlike Newton's laws, which describe the intrinsic laws of nature, Kepler's laws are mainly mathematical summaries of the motions of the planets. It turns out that Kepler's laws are consequences of Newton's laws of motion.

Informally, it states that as each planet moves around the sun, a line drawn from the planet to the sun sweeps out equal areas in equal times. But for the same is visualizing orbits, we state Kepler's first law without proof.

Theorem 2.3.3 (Kepler's First Law)*.* The orbit of a planet is an ellipse with the Sun at one of the two foci.



Theorem 2.3.4 (Kepler's Second Law)*.* Imagine a planet orbiting around the sun fixed at point O. We shall make the approximation that the sun is fixed. It the two pairs of points (P, Q) and (P', Q') are separated by equal time intervals $\Delta t = \Delta t'$, then the two areas ΔA and $\Delta A'$ swept out by the planet are equal.



Proof. Derivation to be done. ■

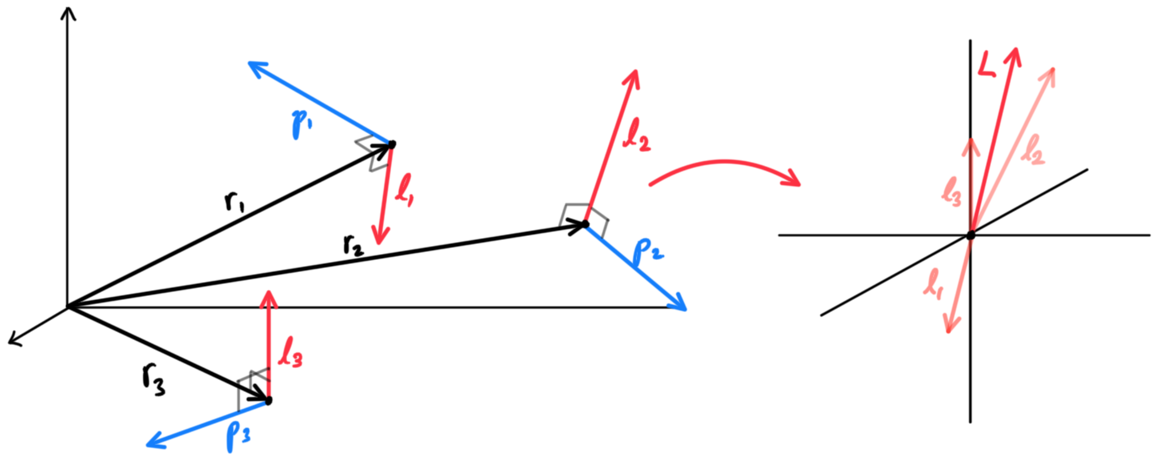
2.3.2 Angular Momentum of Several Particles

We define the angular momentum of a system of particles as the sum of the individual angular momenta of each particle.

Definition 2.3.4 (Total Angular Momentum)**.** Given a system of n particles $\boldsymbol{r_1}, \boldsymbol{r_2}, \ldots, \boldsymbol{r_n}$ with respective angular momenta $\boldsymbol{l_1}, \boldsymbol{l_2}, \ldots, \boldsymbol{l_n}$ around an origin O, the total angular momentum is defined

$$\boldsymbol{L} \equiv \sum_i \boldsymbol{l_i} \equiv \sum_i \boldsymbol{r_i} \times \boldsymbol{p_i}$$

For three particles, we have



Unlike the total linear momentum, this interpretation of the total angular momentum as the sum of its individual angular momenta is **not equivalent** to that being the angular momentum of its center of mass $\boldsymbol{R}$. It is slightly more complicated than that

Lemma 2.3.5 (Equivalent Interpretation of Total Angular Momentum)*.* Given a system of n particles $\boldsymbol{r_1}, \boldsymbol{r_2}, \ldots, \boldsymbol{r_n}$ with respective angular momenta $\boldsymbol{l_1}, \boldsymbol{l_2}, \ldots, \boldsymbol{l_n}$ around an origin O, let its center of mass be $\boldsymbol{R}$, total velocity be $\boldsymbol{V}$, and its total linear momentum be $\boldsymbol{P}$. Then,

$$\boldsymbol{L} = \boldsymbol{R} \times \boldsymbol{P} + \sum_i (\boldsymbol{r_i} - \boldsymbol{R}) \times (\boldsymbol{p_i} - m_i \boldsymbol{V})$$

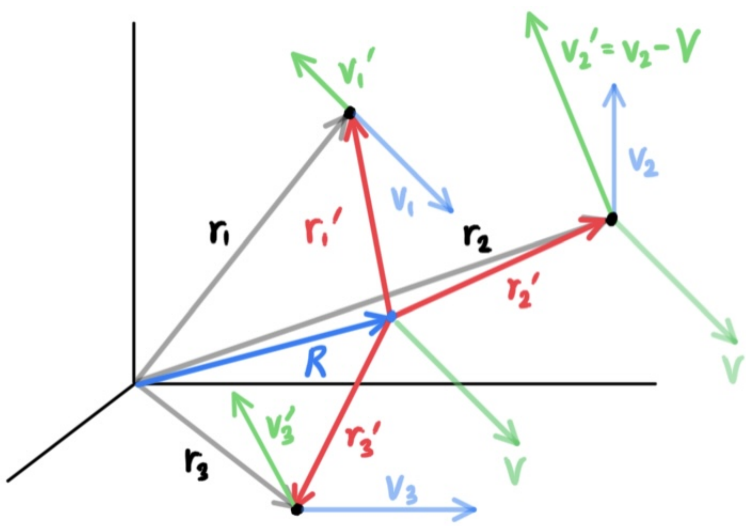
This formula is quite complicated and can be greatly simplified when we consider each vector $\boldsymbol{r_i}$ as the sum of the center of mass $\boldsymbol{R}$ plus another vector $\boldsymbol{r}_i'$. Then, we have

$$\boldsymbol{r_i} = \boldsymbol{R} + \boldsymbol{r}_i'$$

and

$$\boldsymbol{v_i} = \boldsymbol{V} + \boldsymbol{v}_i'$$

where $\boldsymbol{V} \equiv \frac{d\boldsymbol{R}}{dt}$ and $\boldsymbol{v}_i'$ is the velocity of the ith particle relative to the center of mass of the system.



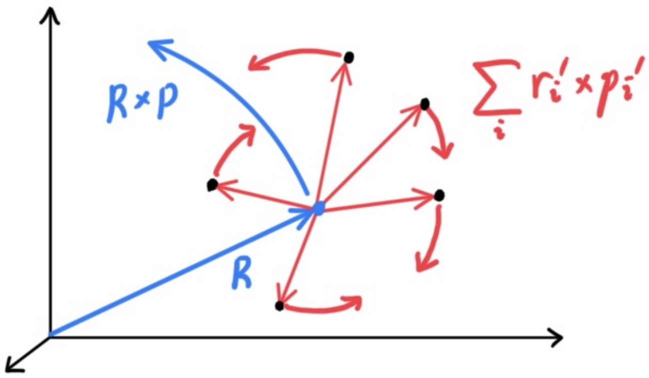
Then, we can calculate

$$\begin{aligned}
\boldsymbol{L} &\equiv \sum_i \boldsymbol{r_i} \times \boldsymbol{p_i} \\
&= \sum_i (\boldsymbol{R} + \boldsymbol{r}'_{\boldsymbol{i}}) \times \big(m_i(\boldsymbol{V} + \boldsymbol{v}'_{\boldsymbol{i}})\big) \\
&= \sum_i \boldsymbol{R} \times m_i \boldsymbol{V} + \sum_i \boldsymbol{r}'_{\boldsymbol{i}} \times m_i \boldsymbol{v}'_{\boldsymbol{i}} + \left(\sum_i m_i \boldsymbol{r}'_{\boldsymbol{i}}\right) \times \boldsymbol{V} + \boldsymbol{R} \times \left(\sum_i m_i \boldsymbol{v}'_{\boldsymbol{i}}\right)
\end{aligned}$$

But since $\sum_i m_i \boldsymbol{r}'_{\boldsymbol{i}}$ defines the radius vectors from the center of mass $\boldsymbol{R}$, the sum of all of them vanishes, meaning that the last two terms of the last expression also vanishes. Therefore,

$$\begin{aligned}
\boldsymbol{L} &= \sum_i \boldsymbol{R} \times m_i \boldsymbol{V} + \sum_i \boldsymbol{r}'_{\boldsymbol{i}} \times m_i \boldsymbol{v}'_{\boldsymbol{i}} \\
&= \boldsymbol{R} \times \boldsymbol{P} + \sum_i \boldsymbol{r}'_{\boldsymbol{i}} \times \boldsymbol{p}'_{\boldsymbol{i}}
\end{aligned}$$

where $\boldsymbol{p}'_{\boldsymbol{i}}$ is the linear momentum of the ith particle with respect to the origin $O = \boldsymbol{R}$. In words, this lemma says that the total angular momentum about a point O is the angular momentum of motion concentrated at the center of mass, *plus* the angular momentum of motion about the center of mass.



Law of Conservation of Angular Momentum

Definition 2.3.5 (Total Torque)**.** We naturally define the *total torque* as the derivative of the total angular momentum. Since the capital gamma letter already represents the torque of a single particle, we just represent total torque as

$$\dot{\boldsymbol{L}} \equiv \sum_i \dot{\boldsymbol{l}}_{\boldsymbol{i}} \equiv \sum_i \boldsymbol{\Gamma} \equiv \sum_i \boldsymbol{r_i} \times \boldsymbol{F}_i$$

This means that the rate of change of $\boldsymbol{L}$ is just the net torque on the whole system.

Given particle α, we can divide up the forces on α into its internal and external forces, as we did before. The net force on particle α is

$$\boldsymbol{F_\alpha} \equiv \sum_{\beta \neq \alpha} \boldsymbol{F}_{\alpha\beta} + \boldsymbol{F}^{ext}_{\boldsymbol{\alpha}}$$

where, again, $\boldsymbol{F_{\alpha\beta}}$ is the force exerted on particle α by particle β, and $\boldsymbol{F_{\alpha}^{ext}}$ is the net force exerted on particle α by all agents outside the n particle system. Then, we can calculate

$$\begin{aligned}
\dot{\boldsymbol{L}} &= \sum_{\alpha} \boldsymbol{r_\alpha} \times \boldsymbol{F_\alpha} \\
&= \sum_{\alpha} \sum_{\beta \neq \alpha} \boldsymbol{r_\alpha} \times \boldsymbol{F_{\alpha\beta}} + \sum_{\alpha} \boldsymbol{r_\alpha} \times \boldsymbol{F_{\alpha}^{ext}} \\
&= \sum_{\alpha} \sum_{\beta > \alpha} (\boldsymbol{r_\alpha} \times \boldsymbol{F_{\alpha\beta}} + \boldsymbol{r_\beta} \times \boldsymbol{F_{\beta\alpha}}) + \sum_{\alpha} \boldsymbol{r_\alpha} \times \boldsymbol{F_{\alpha}^{ext}} \\
&= \sum_{\alpha} \sum_{\beta > \alpha} (\boldsymbol{r_\alpha} - \boldsymbol{r_\beta}) \times \boldsymbol{F_{\alpha\beta}} + \sum_{\alpha} \boldsymbol{r_\alpha} \times \boldsymbol{F_{\alpha}^{ext}}
\end{aligned}$$

where we used the fact that $\boldsymbol{F_{\alpha\beta}} = -\boldsymbol{F_{\beta\alpha}}$. Using the fact that each force $\boldsymbol{F_{\alpha\beta}}$ acts in a straight line (that is, it is *central*), we see that $\boldsymbol{r_\alpha} - \boldsymbol{r_\beta}$ is parallel to $\boldsymbol{F_{\alpha\beta}}$ and so the double sum vanishes.So,

$$\dot{\boldsymbol{L}} = \sum_{\alpha} \boldsymbol{r_\alpha} \times \boldsymbol{F_{\alpha}^{ext}}$$

Now we are ready to present the law.

Theorem 2.3.6 (Law of Conservation of Angular Momentum). The total torque on a system of particles is just equal to the net external torque.

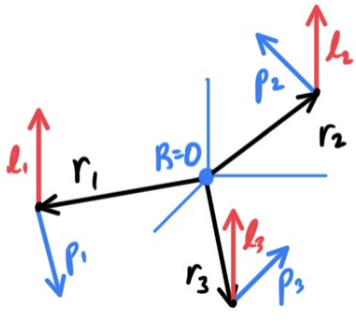
$$\dot{\boldsymbol{L}} = \boldsymbol{\Gamma}^{ext} \equiv \sum_{i} \boldsymbol{r_i} \times \boldsymbol{F_{i}^{ext}}$$

In particular, if the external torque on an n-particle system is zero, then the total angular momentum $\boldsymbol{L}$ is constant.

Theorem 2.3.7 (Angular Momentum about the Center of Mass). The law and the more general result $\dot{\boldsymbol{L}} = \boldsymbol{\Gamma}^{ext}$ were derived on the assumption that all quantities were measured in an inertial frame, so that Newton's second law could be invoked. This required both $\boldsymbol{L}$ and $\boldsymbol{\Gamma}^{ext}$ be measured about an origin O fixed in some inertial frame. Remarkably, the same two results hold if $\boldsymbol{L}$ and $\boldsymbol{\Gamma}^{ext}$ are measured about the center of mass. That is,

$$\frac{d}{dt}\boldsymbol{L}(\text{about CM}) = \boldsymbol{\Gamma}^{ext}(\text{about CM})$$

and hence, if the external torque about the center of mass is 0, then the angular momentum about the center of mass is conserved.



What this means is that if there is no external rotational force (with respect to CM) on the system of particles, then the sum of the individual angular momenta of each particle (about the CM) will be the same, meaning that

$$\boldsymbol{l_1} + \boldsymbol{l_2} + \boldsymbol{l_3} = \text{ constant } C$$

Definition 2.3.6 (Moment of Inertia)**.** The *moment of inertia*, also known as the *angular mass*, of a rigid body is a quantity that determines the torque needed for a desired angular acceleration. It is the equivalent to how mass determines the force needed for a desired linear acceleration.

2.4 Energy

2.4.1 Kinetic Energy and Work

Definition 2.4.1 (Kinetic Energy)**.** The *kinetic energy* of a single particle of mass m traveling with speed v is defined to be

$$T = \frac{1}{2}mv^2$$

Given that the particle is moving through space, we can use the fact that $v^2 = \boldsymbol{v} \cdot \boldsymbol{v}$ and evaluate

$$\frac{dT}{dt} = m\dot{\boldsymbol{v}} \cdot \boldsymbol{v} = \boldsymbol{F} \cdot \boldsymbol{v}$$

Note that both sides are scalars. Using analysis, we can interpret this statement as the following. For a certain time t and $t + \Delta t$, we have

$$T(t + \Delta t) - T(t) = (\boldsymbol{F} \cdot \boldsymbol{v})(\Delta t) + o(\Delta t)$$

which means that for an infinitesimal time dt passed, the change in kinetic energy dT is gets asymptotically close (and therefore, can be interpreted as being equal) to $(\boldsymbol{F} \cdot \boldsymbol{v})dt$.

$$dT = \boldsymbol{F} \cdot \boldsymbol{v}\, dt$$

However, on an infinitesimal scale, we can see that $\boldsymbol{v}dt$ is the best linear approximation for $\boldsymbol{r}$, so $\boldsymbol{r} = \boldsymbol{v}\, dt$. This leads to

$$dT = \boldsymbol{F} \cdot d\boldsymbol{r}$$

We can construct an infinite sum of these values across a certain time period, which converges, by definition, to the value of a Riemann integral.

Definition 2.4.2 (Work-Energy Theorem)**.** Given a particle moving in a path $t \rightarrow \boldsymbol{r}(t) \in \mathbb{R}^3$ within a force field $\boldsymbol{F}$ that defines a force vector for every point on the path, the *work* done by $\boldsymbol{F}$ moving from point 1 to point 2 is

$$W(1 \rightarrow 2) \equiv \Delta T \equiv T_2 - T_2 = \int_1^2 \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{t_1}^{t_2} \boldsymbol{F}(t) \cdot \boldsymbol{v}(t)\, dt$$

Note that $\boldsymbol{F}$ may be a vector field, but there can exist additional parameters on it. For example, $\boldsymbol{F}$ may be dependent on not only the position of the particle in $\mathbb{R}^3$, but also its velocity.

1. Gravitational force gives a force field that is only dependent on the position of a particle from some mass.
2. Drag force, however, is not just dependent on position. It can be (linearly or quadratically) proportional to the velocity of the particle.

We can interpret work as the following: Given a particle that travels along a certain path in $\mathbb{R}^3$ we can partition the path into N intervals I_i with a point chosen in each interval $\boldsymbol{r_i} \in I_i$. This allows us to approximate the integral as

$$\int_1^2 \boldsymbol{F} \cdot d\boldsymbol{r} \approx \sum_{i=1}^N \boldsymbol{F}_{r_i} \cdot \boldsymbol{v}_{r_i}$$

where $\boldsymbol{F}_{r_i}, \boldsymbol{v}_{r_i}$ denotes the force and velocity at $\boldsymbol{r_i}$. At each term of the sum, we can see that

1. if the force $\boldsymbol{F_i}$ acting at that point points in the direction of $\boldsymbol{v_i}$ (i.e. $\boldsymbol{F_i} \cdot \boldsymbol{v_i} > 0$, then this adds a positive value of $\boldsymbol{F_i} \cdot \boldsymbol{v_i}$ to the series. This makes sense since $\boldsymbol{F_i}$ pointing in the same direction as $\boldsymbol{v_i}$ means that the velocity is increasing (and therefore the kinetic energy).
2. if the force $\boldsymbol{F_i}$ acting at that point points in the opposite direction of $\boldsymbol{v_i}$ (i.e. $\boldsymbol{F_i} \cdot \boldsymbol{v_i} < 0$, then this adds a negative value of $\boldsymbol{F_i} \cdot \boldsymbol{v_i}$ to the series. This makes sense since $\boldsymbol{F_i}$ pointing in the opposite direction as $\boldsymbol{v_i}$ means that the velocity is decreasing (and therefore the kinetic energy).
3. if the force $\boldsymbol{F_i}$ acts orthogonally in the direction of $\boldsymbol{v_i}$ (i.e. $\boldsymbol{F_i} \cdot \boldsymbol{v_i} > 0$, then the force contributes nothing in increasing or decreasing the velocity of the particle in the direction.

Setting $N \rightarrow \infty$ (or more specifically, the partition mesh $\lambda \rightarrow 0$) converges the sum onto the value of the integral.

Note that the force $\boldsymbol{F}$ is really the net force of all forces acting on that particle.

$$\boldsymbol{F} = \sum_j \boldsymbol{F}_j$$

For example, the net force on a projectile is the sum of two forces: the weight and drag force. So, when computing the work done by the net force $\boldsymbol{F}$, we can just sum up the work done by the individual forces.

$$\begin{aligned}
W(1 \rightarrow 2) &= \int_1^2 \boldsymbol{F} \cdot d\boldsymbol{r} \\
&= \int_1^2 \sum_j \boldsymbol{F_j} \cdot d\boldsymbol{r} \\
&= \sum_j \int_1^2 \boldsymbol{F_j} \cdot d\boldsymbol{r} = \sum_j W_j(1 \rightarrow 2)
\end{aligned}$$

Therefore, we can write the work energy theorem in the following way:

$$T_2 - T_1 = \sum_{i=1}^n W_i(1 \rightarrow 2)$$

2.4.2 Potential Energy and Conservative Forces

Definition 2.4.3 (Conservative Forces)**.** A conservative force is a time-independent force field $\boldsymbol{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ that is conservative. This means that it must satisfy one of the properties:

1. $\boldsymbol{F}$ is the gradient of some *potential* scalar vector field U.
$$\boldsymbol{F} = -\nabla U$$
2. The curl of $\boldsymbol{F}$ vanishes.
$$\nabla \times \boldsymbol{F} = 0$$
3. There is zero net work done by the force when moving a particle through a closed loop C.
$$W(1 \to 1) \equiv \oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$$
4. The net work done by the force from point 1 to point 2 is path independent. Given two path functions C_1, C_2 from point 1 to point 2, we have
$$\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} = U(1) - U(2)$$

Due to the invariance of work over paths in a conservative force field, we can simplify the formula for work as

$$W(1 \to 2) \equiv \int_1^2 \boldsymbol{F}(t) \cdot \boldsymbol{v}\, dt \equiv \int_1^2 (\boldsymbol{F} \cdot \boldsymbol{v})(t)\, dt$$

where $\boldsymbol{F} \cdot \boldsymbol{v}$ is a scalar function of t. If $\boldsymbol{F}$ is constant (immediately implying it being conservative), we have

$$W(1 \to 2) = \boldsymbol{F} \cdot \boldsymbol{d}$$

where $\boldsymbol{d}$ is the displacement vector from 1 to 2.

Example 2.4.1. *Gravitational and electromagnetic forces are examples of conservative forces. Frictional and drag forces are not conservative.*

Definition 2.4.4 (Potential Energy)**.** Given a conservative force field $\boldsymbol{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, the *potential* is a scalar field $U : \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that

$$\boldsymbol{F} = -\nabla U$$

This means that given

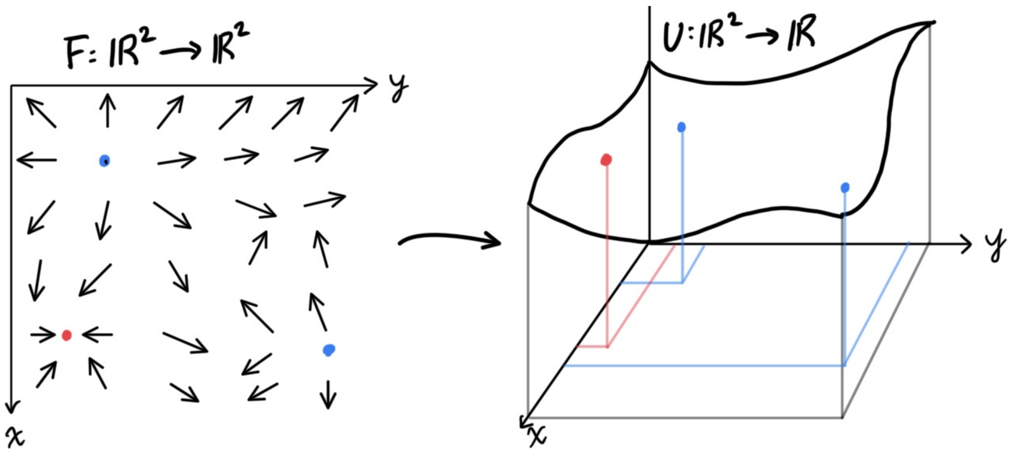
$$\boldsymbol{F} = \nabla U = \begin{pmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \end{pmatrix}$$

the work along a curve C from point 1 to point 2 is

$$W = \int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C -\nabla U \cdot d\boldsymbol{r} = U(1) - U(2)$$

using the gradient theorem for line integrals.

Definition 2.4.5 (Equilibrium Points)**.** We can see that the potential energy is 0 at a point if and only if the force field $\boldsymbol{F} = 0$. This point is called an *equilibrium point*. For the sake is visualization, we construct a conservative force field $\boldsymbol{F}$ in $\mathbb{R}^2$ and its corresponding potential (scalar field) U. We can see that $\boldsymbol{F} = \boldsymbol{0}$ at the local minimum A and $\boldsymbol{F} = \boldsymbol{0}$ at the local maximum B.



The force $\boldsymbol{F}$ tends to push a particle "downhill" towards the minima. Notice that A is a stable equilibrium point, while B is an unstable one (since small perturbations on a particle at B pushes it off).

Lemma 2.4.1 (Conservation of Energy for One Force Acting on One Particle)*.* For conservative vector fields $\boldsymbol{F}$, we see that

$$W = T(2) - T(1) = U(1) - U(2)$$

which implies that for any two points 1 and 2 in $\mathbb{R}^3$, the *mechanical energy (ME)*, defined as the sum of the kinetic energy (KE) and potential energy (PE), always stays constant.

$$T(1) + U(1) = T(2) + U(2) \iff ME = KE + PE \text{ is constant}$$

That is, mechanical energy is conserved.

Theorem 2.4.2 (Law of Conservation of Energy for One Particle)*.* If all of the n forces $\boldsymbol{F}_i$ acting on a particle are conservative, each with its corresponding potential energy $U_i(\boldsymbol{r})$, then the *total mechanical energy*,defined as

$$ME = T + U \equiv T + \sum_i U_i(\boldsymbol{r})$$

is conserved, i.e. constant in time.

Therefore, for nonconservative force fields, we can see that mechanical energy is usually not conserved, meaning that as a particle travels from 1 to 2, energy is either gained or lost (by some external source).

Nonconservative Forces

If some of the forces on our particle are nonconservative, then we cannot define corresponding potential energies nor a conserved mechanical energy. However, we can define potential energies for all forces that are conservative, and then use the Work KE theorem in a form that shows how the nonconservative forces change the particle's mechanical energy.

We divide the net force on the particle into two parts: the conservative part $\boldsymbol{F}_{cons}$ and the nonconservative part $\boldsymbol{F}_{nc}$. Clearly for the conservative part there exists a potential scalar field U associated with $\boldsymbol{F}_{cons}$, where $\boldsymbol{F}_{cons} = -\nabla U$. By the Work-KE theorem, the change in kinetic energy between any two times is

$$\Delta T = W = W_{cons} + W_{nc}$$

but $W_{cons} = -\Delta U$, and so we have $\Delta(T + U) = W_{nc}$. If we define mechanical energy as $E = T + U$, then we have

$$\Delta E \equiv \Delta(T + U) = W_{nc}$$

Mechanical energy is no longer conserved, but we have shown that ME changes exactly to the extent that the nonconservative forces do work on our particle.

Time Dependent Force Fields

In certain cases, a force field $\boldsymbol{F}$ may change with respect to time, meaning that we must interpret as a function

$$F : \mathbb{R}^3 \times T \subset \mathbb{R} \longrightarrow \mathbb{R}^3$$

This also induces the modification of U as

$$U : \mathbb{R}^3 \times T \longrightarrow \mathbb{R}$$

which now has another parameter t, where

$$\boldsymbol{F}(\boldsymbol{r}, t) = -\nabla U(\boldsymbol{r}, t)$$

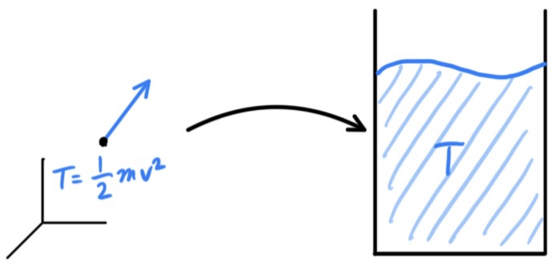
If $\boldsymbol{F}$ is conservative (that is, it stays conservative as t varies), it follows that

$$U_1(t) - U_2(t)$$

stays constant for a constant t. However, since every path function is paramaterized over a time period of positive length, the work done by the particle as it moves from point 1 to point 2 is no longer simply the difference in the function U between the two points. Therefore, even though the total energy $T + V$ may still be defined, it is not conserved.

2.4.3 Summary

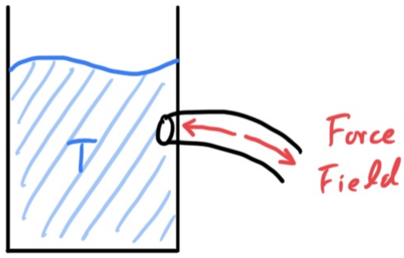
Let us summarize the relationship between the concepts we have talked about in this section. For a particle in a certain system with forces acting on it, we can always define its kinetic energy $T = \frac{1}{2}mv^2$. Let's visualize this by imagining the particle being a "tub" filled with water that represents kinetic energy.



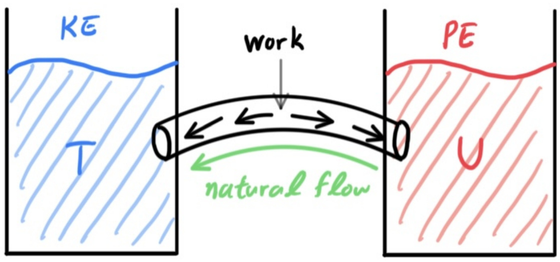
As the particle moves through $\mathbb{R}^3$, the force field must affect the velocity of the particle in some way, whether it slows it down (loses KE) or speeds it up (gains KE). We call this change in kinetic energy from time 1 to time 2 done by $\boldsymbol{F}$ the *work* done by $\boldsymbol{F}$, and the Work-Energy theorem tells us that we can compute this work by summing up the infinitesimal changes in work dT, which is equal to the infinitesimal changes in KE $\boldsymbol{F} \cdot \boldsymbol{v} \, dt$, represented by the line integral

$$\Delta T = \int_1^2 \boldsymbol{F} \cdot d\boldsymbol{r}$$

Going back to the tub analogy, we can think of the force field $\boldsymbol{F}$ as a tube that either adds or sucks away water (KE) from the tub. The rate at which the water is being added/subtracted is determined by the force field, or more specifically by the value of $(\boldsymbol{F} \cdot \boldsymbol{v})(t)$. Therefore, by the fundamental theorem of calculus, we can Riemann integrate this value over time to get the formula above.



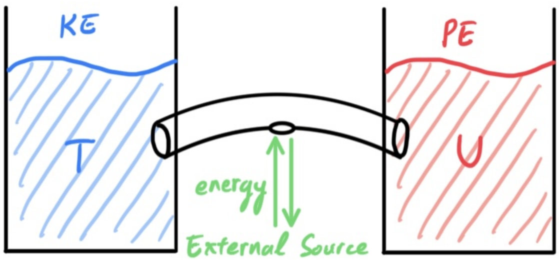
The next natural question would be: where is this external source of water coming from and leaking out to? This is where we split up the case into conservative and nonconservative force fields. If we are working in a conservative force field, we can accordingly define a scalar potential energy field U that tells us how much potential energy this particle has in addition to the kinetic energy. We represent this with a second tub.



It turns out within a conservative vector field, water is transferred between the KE tub and the PE tub and those two tubs only, and generally, the system naturally evolves such that the water from the PE tub flows into the KE tub, but the total amount of water in the two tubs never change. The amount of water transferred between the tubs is known as *work*. Note that

1. water transferred from PE $\to$ KE is positive work
2. water transferred from KE $\to$ PE is negative work

However, in a nonconservative vector field, things are different. Again, the kinetic energy is well-defined, but now there is no scalar potential energy field U that exists. This implies that there are actually leaks in the work tube where energy can come in and out from some other source. Therefore, this can interrupt the flow



For example, let us consider a particle with initial velocity $\boldsymbol{v_0}$ traveling in a 2-dimensional gravitational field starting at (x_0, y_0). Clearly, the gravitational field is time-independent, defined

$$\boldsymbol{F} \equiv \begin{pmatrix} 0 \\ -mg \end{pmatrix}$$

and since it is conservative, we can define the potential scalar field

$$U(x, y) = mgy$$

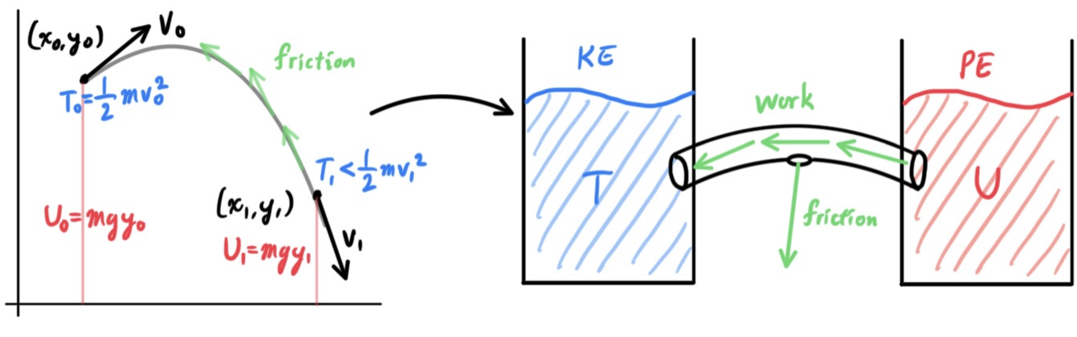
This is equivalent to the high school formula for gravitational potential energy $PE = mgh$. Notice that this is simply the antiderivative of $\boldsymbol{F}_y$. Therefore, the particle would initially have a total mechanical energy of

$$ME = \frac{1}{2}mv_0^2 + mgy_0$$

Therefore, at any point in time, if we know the position of the particle, say (x_1, y_1), we can solve

$$\frac{1}{2}mv_0^2 + mgy_0 = \frac{1}{2}mv_1^2 + mgy_1$$

to get the speed (but not velocity) v_1. However, when the particle is exposed to drag force, then some energy is "stolen" by friction, and not 100% of the potential energy will be converted to kinetic energy.



2.4.4 Energy for One-Dimensional Systems

Many interesting problems involve an object that is constrained to move in just one dimension, and the analysis of such problems is much simpler, albeit still insightful, than the general case. We analyze the motion of particles on $\mathbb{R}$ and then generalize this to the motion on one-dimensional manifolds.

Let us consider an object constrained to move along a perfectly straight track, taking it to be the x-axis (still embedded in $\mathbb{R}^3$). The only component of any force $\boldsymbol{F}$ that can do work is the x component F_x, and so the work done by $\boldsymbol{F}$ is the one-dimensional integral

$$W(x_1 \to x_2) = \int_{x_1}^{x_2} F_x(x)\, dx$$

Assuming that $\boldsymbol{F}$ is a conservative vector field, we let U be its potential (i.e. its primitive) defined

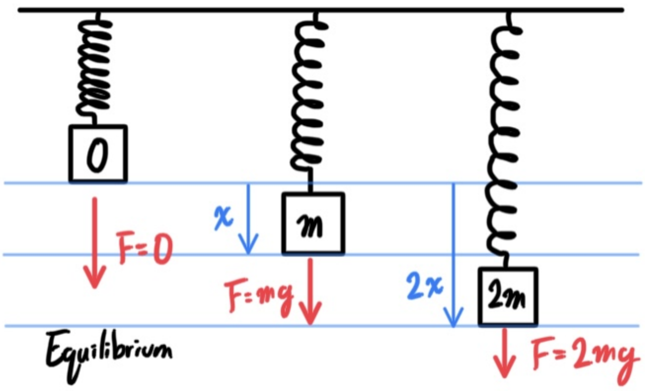
$$U(x) \equiv -\int_{x_0}^{x} F_x(x')\, dx'$$

where x_0 is a reference point such that $U(x_0) = 0$.

Example 2.4.2 (Hooke's Law). *Hooke's law states that the force F needed to extend or compress a spring by some distance x scales linearly with respect to that distance. That is,*

$$F_x = -kx$$

where k is a constant factor characteristic of the spring (i.e. its stiffness), and x is small compared to the total possible deformation of the spring.



For a mass on the end of a spring obeying Hooke's law, the force is $F_x = -kx$, and if we choose the reference point $x_0 = 0$, we have

$$U(x) = -\int_0^x -kx\, dx = \frac{1}{2}kx^2$$

Solution of the Motion

Another feature of one-dimensional conservative systems is that we can use the conservation of energy to obtain a complete solution of the motion. In summary, since $E = T + U(x)$ is conserved, with $U(x)$ a known function and E determined by the initial conditions, we can see that

$$T = \frac{1}{2}mv^2 = E - U(x)$$

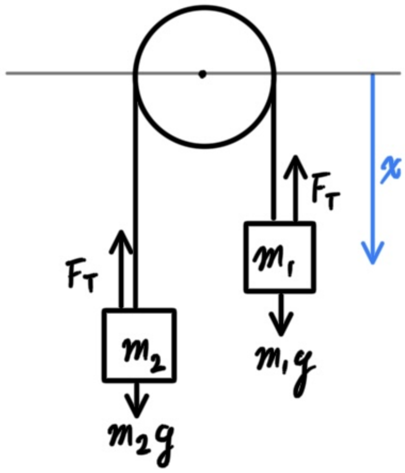
and hence we can write the velocity as a function of x.

$$v(x) = \pm\sqrt{\frac{2}{m}}\sqrt{E - U(x)}$$

Note that there is an ambiguity in the sign since energy considerations cannot determine the direction of the velocity (and so we cannot extend this method to a 3-dimensional system). This is in fact a differential equation that may or may not be solvable.

Curvilinear One-Dimensional Systems

An Atwood machine consists of two masses m_1, m_2 suspended from opposite ends of a massless, inextendible string that passes over a frictionless pulley. The two masses can move up and down, but the forces of the pulley on the string and the string on the masses constrain matters so that the mass m_2 can move up only to the extent that m_1 moves down by exactly the same distance. Therefore, the position of the whole system an be specified by a single parameter, for example the height x of m_1 below the pulley's center.



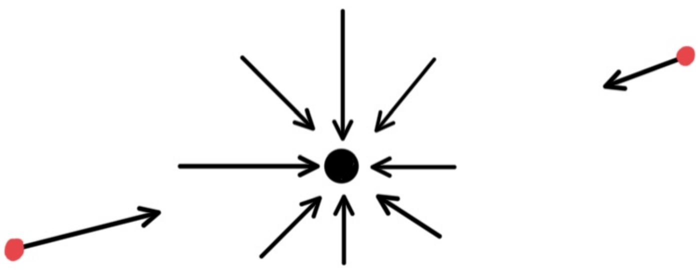
Let us consider the energies of the masses m_1, m_2. The forces acting on them are gravity and the string tension. Since gravity is conservative, we can introduce potential energies U_1, U_2 for the gravitational forces.

2.4.5 Central Forces

Definition 2.4.6 (Central Force)**.** A central force is a force field $\boldsymbol{F}$ that is everywhere directed toward or away from a fixed force center. Taking the force center to be the origin, a central force has the form

$$\boldsymbol{F}(\boldsymbol{r}) = f(\boldsymbol{r})\,\hat{\boldsymbol{r}}$$

where $f(\boldsymbol{r})$ gives the signed magnitude of the force.



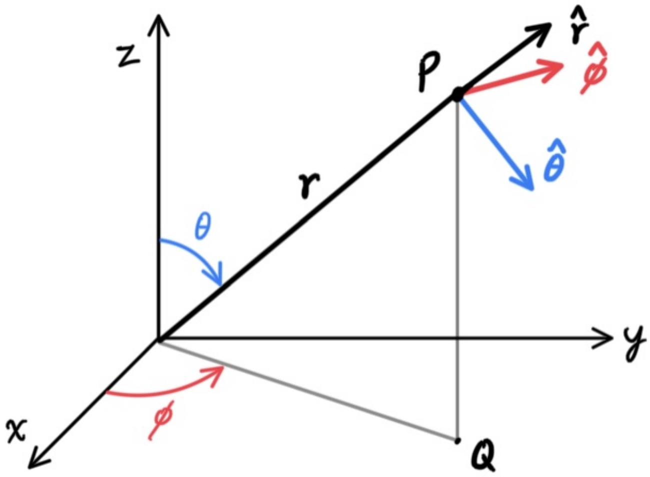
Not all central forces are conservative. However, a central force is conservative if and only if it is spherically symmetric.

Definition 2.4.7 (Spherical Coordinates)**.** When working with central forces, we usually work in spherical coordinates due to the rotational invariance of $\boldsymbol{F}$.

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

The angle ϕ is often called the *azimuth*. Note that like the polar coordinates, the unit vectors $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ are all orthonormal, with

1. $\hat{\boldsymbol{r}}$ pointing in the direction of $\boldsymbol{r}$.
2. $\hat{\boldsymbol{\theta}}$ pointing in the direction of increasing θ with r, ϕ fixed.
3. $\hat{\boldsymbol{\phi}}$ pointing in the direction of increasing ϕ with r, θ fixed.



The statement that a function $f(\boldsymbol{r})$ is spherically symmetric is simply the statement that, with $\boldsymbol{r}$ expressed in spherical polars, f is independent of θ and ϕ. That is,

$$\frac{\partial f}{\partial \theta} \text{ and } \frac{\partial f}{\partial \phi}$$

are zero everywhere.

We know that the gradient of a scalar function f is

$$\nabla f = \hat{\boldsymbol{x}}\frac{\partial f}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial f}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial f}{\partial z}$$

Using a bit of analysis, we get the following theorem.

Theorem 2.4.3 (Gradient in Polar Coordinates). Given a function f of three variables r, θ, ϕ, an infinitesimal change in f is

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi}d\phi$$

Some computation leads to the gradient of f being

$$\nabla f = \hat{\boldsymbol{r}}\frac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

Therefore, given that a central force $\boldsymbol{F}(\boldsymbol{r})$ is conservative, if can be expressed in the form $-\nabla U$, which has the form

$$\boldsymbol{F}(\boldsymbol{r}) = -\nabla U = -\hat{\boldsymbol{r}}\,\frac{\partial U}{\partial r} - \hat{\boldsymbol{\theta}}\,\frac{1}{r}\frac{\partial U}{\partial \theta} - \hat{\boldsymbol{\phi}}\,\frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}$$

but since $\frac{\partial U}{\partial \theta} = \frac{\partial U}{\partial \phi} = 0$, the formula reduces to

$$\boldsymbol{F}(\boldsymbol{r}) = -\hat{\boldsymbol{r}}\,\frac{\partial U}{\partial r}$$

2.4.6 Energy of Interaction of Two Particles

Two Particle System

Suppose there are two particles acting via force $\boldsymbol{F}_{12}$ (on particle 1 by particle 2) and $\boldsymbol{F}_{21}$, with no other external forces. In general the force $\boldsymbol{F}_{12}$ could depend on the positions of both particles, so we can write the forces as

$$\boldsymbol{F}_{12} = \boldsymbol{F}_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2)$$

and by Newton's third law

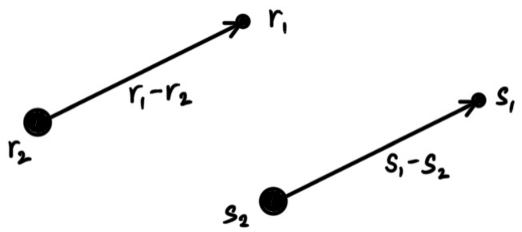
$$\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$$

For any isolated two-particle system, it is known to be translationally invariant (e.g. gravity, electromagnetic force). This means that if we were to shift the entire two-particle system somewhere else, the forces acting upon each other would be the same. That is, given pair of points $\boldsymbol{r}_1, \boldsymbol{r}_2$ and another $\boldsymbol{s}_1, \boldsymbol{s}_2$ such that the distance vectors $\boldsymbol{s}_1 - \boldsymbol{s}_2 = \boldsymbol{r}_1 - \boldsymbol{r}_2$, we have $\boldsymbol{F}_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \boldsymbol{F}_{12}(\boldsymbol{s}_1, \boldsymbol{s}_2)$.

Example 2.4.3 (Gravity). *The gravitational force between two particles would be*

$$\boldsymbol{F}_{12}(\boldsymbol{r}_1 - \boldsymbol{r}_2) = -\frac{Gm_1 m_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3}(\boldsymbol{r}_1 - \boldsymbol{r}_2)$$

which is dependent purely on the difference vector between them.



In other words, $\boldsymbol{F}_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ depends only on $\boldsymbol{r}_1 - \boldsymbol{r}_2$, and so we can write

$$\boldsymbol{F}_{12} = \boldsymbol{F}_{12}(\boldsymbol{r}_1 - \boldsymbol{r}_2)$$

Now, let us fix $\boldsymbol{r}_2$. Assuming that $\boldsymbol{F}_{12}$ is conservative, we can define a potential energy function $U(\boldsymbol{r}_1)$ representing the potential of 1 sitting in the field generated by 2 satisfying

$$\boldsymbol{F}_{12}(\boldsymbol{r}_1 - \boldsymbol{r}_2) = -\nabla_1 U(\boldsymbol{r}_1 - \boldsymbol{r}_2)$$

Remember that $\boldsymbol{r}_2$ is fixed, and so both $\boldsymbol{F}_{12}$ and U_{12} are functions of $\boldsymbol{r}_1$ only!

$$\begin{aligned} \boldsymbol{F}_{12} &\implies \boldsymbol{r}_1 \mapsto \boldsymbol{F}_{12}(\boldsymbol{r}_1 - \boldsymbol{r}_2) \\ U_{12} &\implies \boldsymbol{r}_1 \mapsto U_{12}(\boldsymbol{r}_1 - \boldsymbol{r}_2) \end{aligned}$$

N-particle System

Definition 2.4.8 (Total Kinetic Energy of a System)**.** Assume that there are N particles in a system, labeled $\alpha = 1, 2, \ldots, N$, the *total kinetic energy* is just the sum of the N separate energies.

$$T = \sum_\alpha T_\alpha = \sum_\alpha \frac{1}{2} m_\alpha v_\alpha^2 = \frac{1}{2} \sum_\alpha m_\alpha \dot{\boldsymbol{r}}_\alpha \cdot \dot{\boldsymbol{r}}_\alpha$$

If we decompose the position of each particle as $\boldsymbol{r}_\alpha = \boldsymbol{R} + \boldsymbol{r}'_\alpha$, where $\boldsymbol{R}$ is the center of mass of the system, we have

$$\begin{aligned} T &= \frac{1}{2} \sum_\alpha m_\alpha \left(\dot{\boldsymbol{R}} + \dot{\boldsymbol{r}'}_\alpha \right)^2 \\ &= \frac{1}{2} \sum_\alpha m_\alpha \dot{\boldsymbol{R}}^2 + \dot{\boldsymbol{R}} \cdot \sum_\alpha m_\alpha \dot{\boldsymbol{r}'}_\alpha + \frac{1}{2} \sum_\alpha m_\alpha \dot{\boldsymbol{r}'}^2_\alpha \\ &= \frac{1}{2} M \dot{\boldsymbol{R}}^2 + \frac{1}{2} \sum_\alpha m_\alpha \dot{\boldsymbol{r}'}^2_\alpha \end{aligned}$$

This means that the kinetic energy can be split up into the kinetic energy of the center of mass, together with the kinetic energy of the particles moving around the center of mass.

Theorem 2.4.4 (Work Energy Theorem of a System)*.* Now, assume that for each particle α, it moves along a trajectory $\mathcal{C}_\alpha$. By applying the Work-Energy theorem on each particle and summing, the work is the difference in total kinetic energy, which is

$$\begin{aligned} W(t_1 \to t_2) &= T(t_2) - T(t_1) \\ &= \frac{1}{2} \sum_\alpha m_\alpha \big(v_\alpha(t_2) \big)^2 - \frac{1}{2} \sum_\alpha m_\alpha \big(v_\alpha(t_1) \big)^2 \\ &= \sum_\alpha \int_{\mathcal{C}_\alpha} \boldsymbol{F}_\alpha^{ext} \, d\boldsymbol{r}_\alpha + \sum_\alpha \sum_{\beta \neq \alpha} \int_{\mathcal{C}_\alpha} \boldsymbol{F}_{\alpha\beta} \cdot d\boldsymbol{r}_\alpha \end{aligned}$$

which can also be simplified with the notation

$$W(1 \to 2) = T(t_2) - T(t_1) = \sum_\alpha \int_1^2 \boldsymbol{F}_\alpha^{ext} \cdot d\boldsymbol{r}_\alpha + \sum_\alpha \int_1^2 \boldsymbol{F}_{\alpha\beta} \cdot d\boldsymbol{r}_\alpha$$

where 1 and 2 are not points, but rather *configurations* of the particles of the entire system.

If we would like to define a potential energy, we require that both external and internal forces are conservative.

Definition 2.4.9 (Total Potential Energy of a System)**.** Assume that all forces in a N-particle system are conservative. For each pair of particles $\alpha\beta$, the *potential energy* $U_{\alpha\beta}$ determines the potential of α within the force field generated by β. That is,

$$\boldsymbol{F}_{\alpha\beta}(\boldsymbol{r}_\alpha) = -\nabla_\alpha U_{\alpha\beta}(\boldsymbol{r}_\alpha - \boldsymbol{r}_\beta)$$

where ∇_α represents the del operator with respect to the coordinates of α. Additionally, let U_α^{ext} be the potential of α within the force field generated by the net external force.

$$\boldsymbol{F}_\alpha^{ext}(\boldsymbol{r}_\alpha) = -\nabla_\alpha U_\alpha^{ext}(\boldsymbol{r}_i)$$

Therefore, the total potential energy is

$$U = U^{int} + U^{ext} = \sum_{\alpha} \sum_{\beta > \alpha} U_{\alpha\beta}(\boldsymbol{r}_\alpha - \boldsymbol{r}_\beta) + \sum_{\alpha} U_{\alpha}^{ext}(\boldsymbol{r}_\alpha)$$

2.5 Oscillations

Almost any system that is displaced from a position of stable equilibrium exhibits oscillations. If the displacement is small, the oscillations are almost always of the type called simple harmonic.

2.5.1 Hooke's Law

Theorem 2.5.1 (Hooke's Law). The force exerted by a spring has the form (confined to the x-axis)

$$F_x(x) = -kx$$

where x is the displacement of the spring from its equilibrium length and k is a positive number called the force constant, dependent on the type of spring.

1. If $k > 0$ then the equilibrium at $x = 0$ is stable. We can see this because when $x = 0$ there is no force, when $x > 0$ (displacement to the right) the force is negative (to the left), and when $x < 0$ the force is positive. We call this type of force a *restoring force.*
2. If $k < 0$, the force would be away from the origin, and the equilibrium would be unstable, in which case we do not expect to see oscillations.

By focusing on the x-component, we can see that $F_x(x)$ is a conservative field, and therefore we can define the potential (which is just the antiderivative) as

$$U(x) = \frac{1}{2}kx^2$$

Corollary 2.5.1.1 (General Validity of Hooke's Law for Small Displacements). Consider an arbitrary conservative one-dimensional systsm which is specified by a coordinate x and has potential energy $U(x)$. Let there be a stable equilibrium position $x = x_0$ for which we can let $x_0 = 0$ without loss of generality. Then, assuming that U is analytic, we can write

$$U(x) = U(0) + U'(0)x + \frac{1}{2}U''(0)x^2 + \dots$$

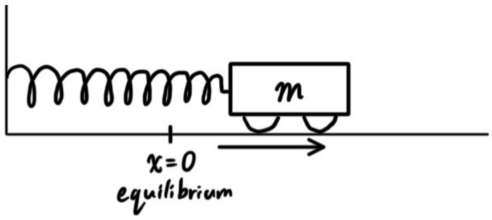
as long as x is small, we can approximate U with the first three terms of the series. The first term is a constant, and since we can always subtract a constant from $U(x)$ without affecting any physics, we may as well define $U(0) = 0$. Because $x = 0$ is an equilibrium point, $U'(0) = 0$ and the second term is automatically 0. Finally, since the equilibrium is stable, it is concave up and therefore $U''(0) > 0$. Renaming $U''(0)$ as k, we can see that for small displacements it is a good approximation to take

$$U(x) = \frac{1}{2}kx^2$$

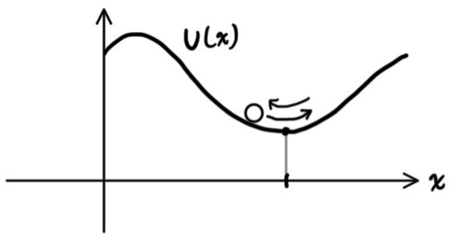
Therefore, for sufficiently small displacements from stable equilibrium, Hooke's law is always valid.

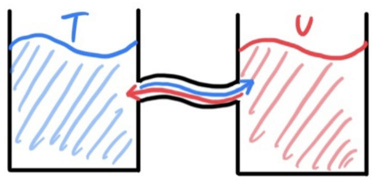
2.5.2 Simple Harmonic Motion

Let us examine the equation of motion for a mass m that is displaced from a position of stable equilibrium. Let us consider a card on a frictionless track attached to a fixed spring, choosing the one-dimensional frame so that the equilibrium position is at $x = 0$.



Note that there are other systems that have the same model as this one, such as a pendulum swinging back and forth. Qualitatively, we can interpret this motion as a particle swinging back and forth around an equilibrium point in a potential scalar field, or using the tub analogy, water flowing back and forth the KE and PE tubs.





Lemma 2.5.2 (Equation of Simple Harmonic Motion)*.* We know that we can approximate the potential energy by Hooke's law to get $F_x(x) = -kx$. Thus the equation of motion is

$$m\ddot{x} = -kx \implies \ddot{x} = -\frac{k}{m}x$$

Let us introduce the constant $\omega = \sqrt{k/m}$ and rewrite the differential equation as

$$\ddot{x} = -\omega^2 x$$

which has general solution

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \iff x(t) = B_1 \cos(\omega t) + B_2 \sin(\omega t)$$

where $B_1 = C_1 + C_2, B_2 = i(C_1 - C_2)$.

Since we are working in the real number line, let us focus on the equation

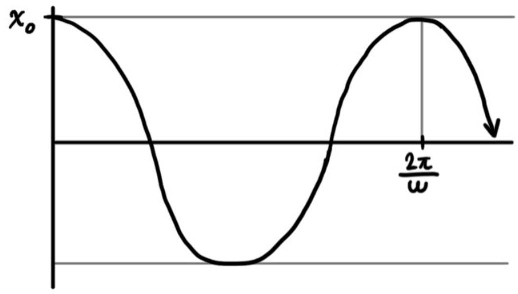
$$x(t) = B_1 \cos(\omega t) + B_2 \sin(\omega t)$$

Simple calculations show that

1. The initial displacement is $x(0) = B_1$
2. The initial velocity is $x'(0) = \omega B_2$

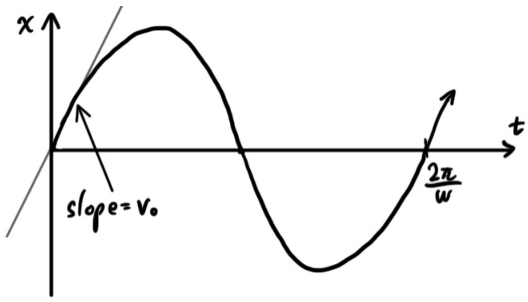
If I start the oscillations by pulling the cart aside to $x = x_0$ and releasing it from rest $v_0 = 0$), then $B_2 = 0$ and only the cosine terms survives, so that

$$x(t) = x_0 \cos(\omega t)$$

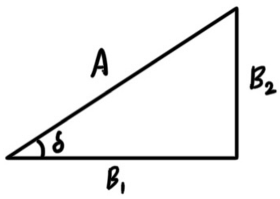


If I launch the cart from the origin ($x_0 = 0$) by giving it a kick at $t = 0$ with velocity of v_0, only the sine term survives and

$$x(t) = \frac{v_0}{\omega} \sin(\omega t)$$



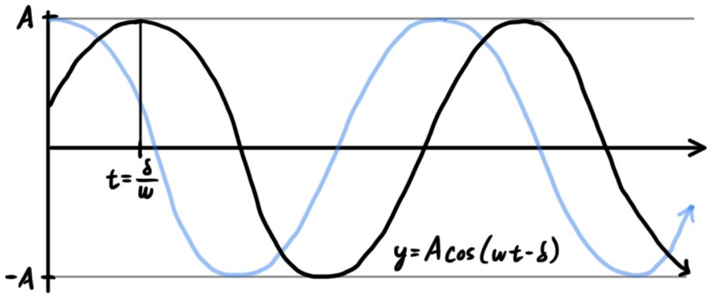
Definition 2.5.1 (Phase-Shifted Cosine Solution)**.** To visualize what happens when a cart has initial displacement x_0 with initial velocity v_0, we first define $A = \sqrt{B_1^2 + B_2^2}$ and $\delta = \arctan(B_1/B_2)$.



We can rewrite the general equation of motion as

$$\begin{aligned} x(t) &= A\left(\frac{B_1}{A}\cos(\omega t) + \frac{B_2}{A}\sin(\omega t)\right) \\ &= A\big(\cos(\delta)\,\cos(\omega t) + \sin(\delta)\,\sin(\omega t)\big) \\ &= A\cos(\omega t - \delta) \end{aligned}$$

This equivalent form shows that the cart is oscillating with amplitude A, but instead of it being a simple cosine, it is a cosine that is horizontally shifted. That is, the oscillations lag behind the simple cosine by the *phase shift* δ.



Energy of Simple Harmonic Oscillators

Since we have worked in a conservative vector field in $\mathbb{R}$ (by neglecting friction), mechanical energy should be conserved, by the law of conservation of energy. Since $x(t) = A\cos(\omega t - \delta)$, the kinetic energy is

$$\begin{aligned} T = \frac{1}{2}m\dot{x}^2 &= \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t - \delta) \\ &= \frac{1}{2}kA^2 \sin^2(\omega t - \delta) \end{aligned}$$

From the force field $F_x(x) = -kx$ and its associated potential field $U = \frac{1}{2}kx^2$, we can see that the potential energy as a function of time is

$$U = \frac{1}{2}k\big(x(t)\big)^2 = \frac{1}{2}kA^2 \cos^2(\omega t - \delta)$$

Adding both gets

$$E = T + U = \frac{1}{2}kA^2$$

which is consistent with our law.

2.5.3 Two-Dimensional Oscillators

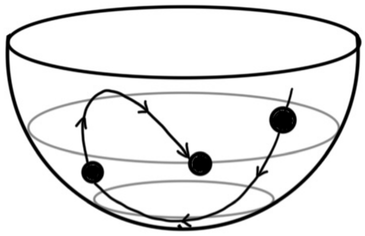
In multiple dimensions, the possibilities for oscillation are considerable richer than in one-dimension.

Definition 2.5.2 (Isotropic Harmonic Oscillator)**.** An *isotropic harmonic oscillator* is an oscillator for which the restoring force is proportional to the displacement from equilibrium, with the same constant of proportionality in all directions.

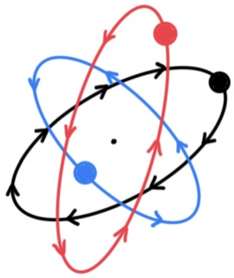
$$\boldsymbol{F} = -k\boldsymbol{r} \implies \begin{cases} F_x = -kx \\ F_y = -ky \end{cases}$$

This is just a generalization of Hooke's law in multiple dimensions. This force is a central force directed towards the equilibrium position, which (without loss of generality) we take to be the origin by choosing an appropriate frame.

Example 2.5.1 (2-Dimensional Isotropic Oscillator)**.** *A two-dimensional isotropic oscillator is (at least approximately) a ball bearing rolling near the bottom of a large spherical bowl.*



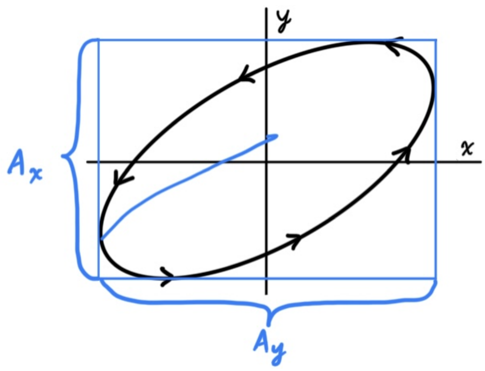
Example 2.5.2 (3-Dimensional Isotropic Oscillator)**.** *A proton or neutron moving within the nucleus of an atom is an example of a three-dimensional isotropic oscillator.*



Lemma 2.5.3 (Equation of Isotropic Harmonic Motion)*.* With the equation of motion $\boldsymbol{F} = m\ddot{\boldsymbol{r}}$, we can see that

$$\begin{cases} \ddot{x} = -\omega^2 x \\ \ddot{y} = -\omega^2 y \end{cases} \implies \begin{cases} x(t) = A_x \cos(\omega t - \delta_x) \\ y(t) = A_y \cos(\omega t - \delta_y) \end{cases}$$

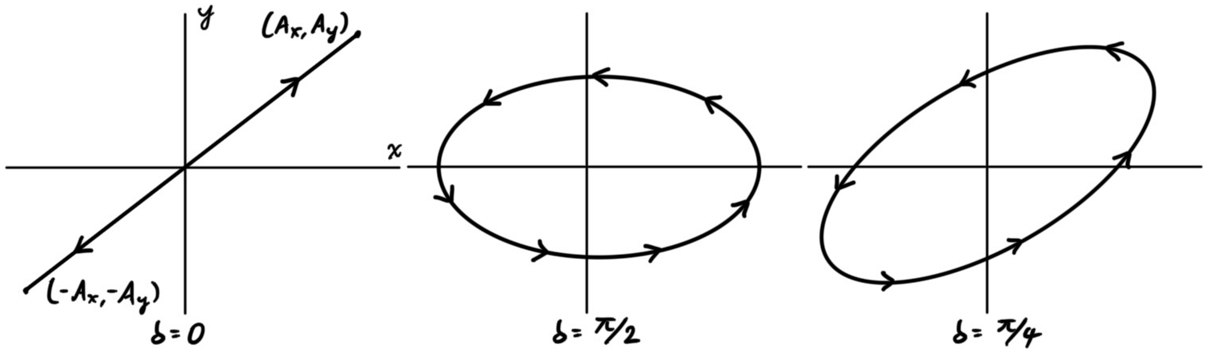
where $\omega = \sqrt{k/m}$, and the four constants $A_x, A_y, \delta_x, \delta_y$ are determined by the initial conditions of the problem. To visualize this, we can see that A_x, A_y determine how "far out" the particle reaches in its oscillation, and the δ_x, δ_y determines the relative shift in oscillation.



However, by redefining the origin of time, we can have the system start when the x-component of the velocity is 0 (that is, when $\delta_x = 0$). However, this does not guarantee that $\delta_y = 0$. Therefore, the simplest form for the general solution is

$$\begin{aligned} x(t) &= A_x \cos(\omega t) \\ y(t) &= A_y \cos(\omega t - \delta) \end{aligned}$$

where $\delta = \delta_y - \delta_x$ is the *relative* phase of the x and y oscillations. This δ value determines the delay (shift) in oscillation between the x and y-components of the two-dimensional oscillating particle. We show the motions of anisotropic oscillators as δ varies. However, note that since both the x and y components have the same frequencies, we are left with relatively simple motions.



1. When $\delta = 0$, then $x(t)$ and $y(t)$ rise and fall in step, moving along the line segment that joins (A_x, A_y) to $(-A_x, -A_y)$.
2. When $\delta = \pi/2$, then x and y oscillate out of step, with x at an extreme when $y = 0$, and vice versa.
3. For other values of δ, the point (x, y) moves in a slanting ellipse.

Definition 2.5.3 (Anisotropic Harmonic Oscillator)**.** In an *anisotropic harmonic oscillator*, the components of the restoring force are proportional to the components of the displacement, but with different constants of proportionality. That is, we have

$$m\ddot{\boldsymbol{x}} = -\begin{pmatrix} k_x \\ k_y \end{pmatrix} \boldsymbol{x} \implies \begin{cases} \ddot{x} = -\omega_x^2 x \\ \ddot{y} = -\omega_y^2 y \end{cases}$$

where there are now different frequencies for the different axes: $\omega_x = \sqrt{k_x/m}, \omega_y = \sqrt{k_y/m}$.

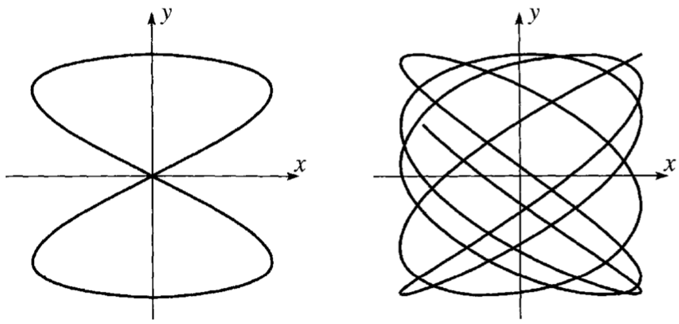
Example 2.5.3 (3-Dimensional Anisotropic Oscillator)**.** *The force felt by an atom displaced from its equilibrium position in a crystal of low symmetry, where is experiences different force constants along the different axis is an anisotropic harmonic oscillator.*

Lemma 2.5.4 (Equation of Anisotropic Harmonic Motion)*.* The solutions to the anisotropic differential equations is a

$$\begin{aligned} x(t) &= A_x \cos(\omega_x t) \\ y(t) &= A_y \cos(\omega_y t - \delta) \end{aligned}$$

Because of the two different frequencies, there is a much richer variety of possible motions.

1. If ω_x/ω_y is rational, then the motion is periodic, resulting in a path called a *Lissajous figure*. In the figure below, we have $\omega_x/\omega_y = 2$ and so the x motion repeats itself twice as often as the y motion.
2. If ω_x/ω_y is irrational, then the motion is called *quasiperiodic*, since the components are periodic but the two periods are incompatible, and so the motion of $\boldsymbol{r}$ is not. That is, when $\omega_x/\omega_y = \sqrt{2}$, we have



(a) $\omega_x = 2\,\omega_y$ (b) $\omega_x = \sqrt{2}\,\omega_y$

2.5.4 Dampened Oscillations

We now introduce resistive forces that will damp the oscillations. There are several possibilities for the resistive force.

1. Ordinary sliding friction is approximately constant in magnitude, but always directed opposite of the velocity.
2. The resistance offered by a fluid, such as air or water, depends on the velocity in a complicated way, but as we saw before, it is a reasonable approximation to assume that the resistive force is proportional to v or v^2.

We will assume that the resistive force is proportional to v.

Definition 2.5.4 (Dampened Simple Harmonic Oscillator)**.** Consider an object in one dimension, such as a cart attacehd to a spring, that is subject to Hooke's force $-kx$ and a resistive force $-b\dot{x}$. The net force on the object is $-b\dot{x} - kx$ and Newton's second law gives

$$m\ddot{x} = -b\dot{x} - kx \implies m\ddot{x} + b\dot{x} + kx = 0$$

which becomes a second degree linear homogeneous differential equation. We divide both sides by m and rewrite the equation as

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \text{ where } \beta = \frac{b}{2m}, \omega_0 = \sqrt{\frac{k}{m}}$$

Note that

1. β is called the *damping constant*, which is a convenient way to characterize the strength of the dampening force (greater β means greater dampening).
2. ω_0 is the systems *natural frequency*, the frequency at which it would oscillate if there were no resistive force present.

Lemma 2.5.5 (Equation of Dampened Harmonic Motion). The general solution to the differential equation of dampened simple harmonic motion is

$$x(t) = e^{-\beta t}\Big(C_1 \exp\big(\sqrt{\beta^2 - \omega_0^2}\, t\big) + C_2 \exp\big(-\sqrt{\beta^2 - \omega_0^2}\, t\big)\Big)$$

Clearly, if there is no damping (i.e. $\beta = 0$), then we are left with the familiar formula for the undampened harmonic oscillator.

$$x(t) = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_t}$$

Weak Damping

This first type of damping is when the damping force is weak in the sense that even though it slows down the particle's motion, the particle still passes the equilibrium point with every oscillation. More specifically, suppose that

$$\beta < \omega_0$$

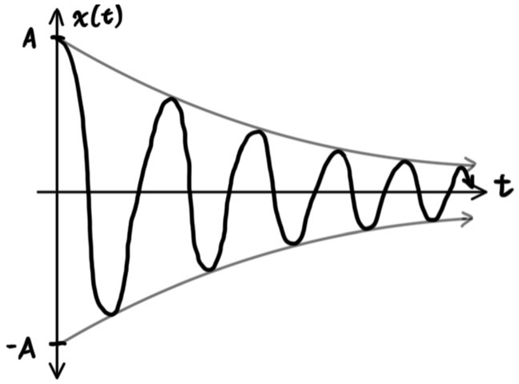
Then, we can write

$$\sqrt{\beta^2 - \omega_0^2} = i\sqrt{\omega_0^2 - \beta^2} = i\omega_1 \text{ where } \omega_1 = \sqrt{\omega_0^2 - \beta^2}$$

The parameter ω_1 is a frequency that is less than the natural frequency ω_0, and with this notation the solution becomes

$$\begin{aligned} x(t) &= e^{-\beta t}\big(C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}\big) \\ &= e^{-\beta t} \cdot A\cos(\omega_1 t - \delta) \end{aligned}$$

which is really just the product of a harmonic function with an exponentially decaying function. The graph of the solution would look like



where for larger β the more rapidly the oscillations die out.

Strong Damping

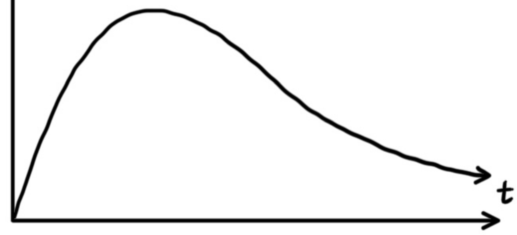
Strong damping happens when the motion is so damped that it completes no oscillations. When the dampening constant β is large, i.e.

$$\beta > \omega_0$$

our solution of motion becomes

$$x(t) = C_1 \exp\left(- \left(\beta - \sqrt{\beta^2 - \omega_0^2}\right) t\right) + C_2 \exp\left(- \left(\beta + \sqrt{\beta^2 - \omega_0^2}\right) t\right)$$

which results in the following graph.



Note that as an oscillator was kicked from the origin at $t = 0$, it slid out to a maximum displacement and then slid ever more slowly back again, returning to the origin only in the limit $t \to +\infty$.

Critical Damping

The boundary between weak and strong damping is called *critical damping*, when

$$\beta = \omega_0$$

This leads to there being a double root in the characteristic polynomial of the differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \implies \ddot{x} + 2\beta\dot{x} + \beta^2 x = 0$$

and by variation of constants, we can see that the general solution is of the form

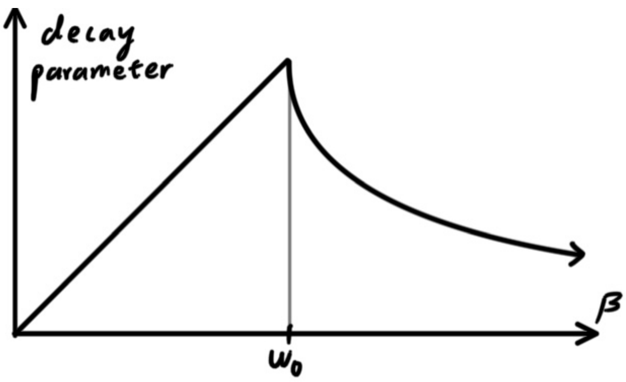
$$\begin{aligned} x(t) &= C_1 e^{-\beta t} + C_2 t e^{-\beta t} \\ &= e^{-\beta t}\big(C_1 + C_2 t\big) \end{aligned}$$

Since the $e^{-\beta t}$ factor dominates the decay of oscillations as $t \to \infty$, we can say that $x(t)$ decays at about the rate of $e^{-\beta t}$.

Comparing all three, notice that

1. Oscillations with weak damping decay exponentially at the rate of $e^{-\beta t} \implies$ decay parameter $= \beta$
2. Oscillations with critical damping decay exponentially at the rate of $e^{-\beta t} \implies$ decay parameter $= \beta$
3. Oscillations with strong damping decay exponentially at the rate of $e^{\beta - \sqrt{\beta^2 - \omega_0^2}} \implies$ decay parameter $= \beta - \sqrt{\beta^2 - \omega_0^2}$

It follows that oscillations die out most quickly when damping is critical, as seen in the graph that plots the decay parameter with respect to the damping constant.



2.5.5 Driven Damped Oscillations

Any natural oscillator would come to rest due to the damping forces. Therefore, in order for the oscillations to continue, one must arrange some external "driving" force to maintain them.

Definition 2.5.5 (External Driving Force)**.** If we denote the external driving force by $F(t)$ and we if assume as before that the damping force has the form $-bv$, then the net force on the oscillator is $\boldsymbol{F}(t) = -bv(t) - kx(t) + F(t)$, resulting in the differential equation

$$m\ddot{x} + b\dot{x} + kx = F$$

which can be rewritten

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)$$

where $f(t) = F(t)/m, \beta = \frac{b}{2m}, \omega_0^2 = k/m$. This is clearly a inhomogeneous linear differential equation.

Now, we specialize to the case that the driving force $f(t)$ is a sinusoidal function of time.

$$f(t) = f_0 \cos(\omega t)$$

Therefore, we are left with solving the differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$$

which has solution

$$x(t) = A\cos(\omega t - \delta) + C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

where

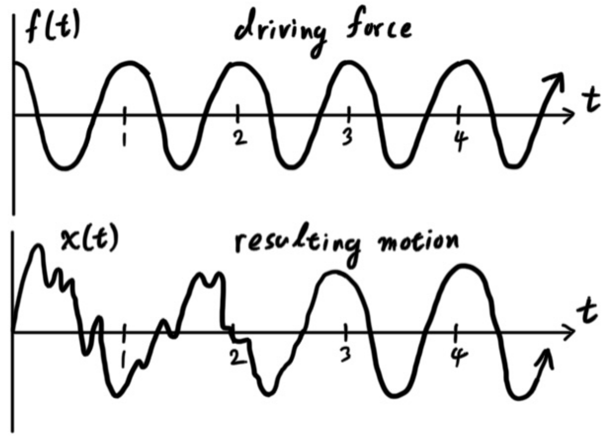
$$A = \sqrt{\frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \text{ and } \delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

Note that the exponential terms, called *transient terms*, are the solutions of the corresponding homogeneous differential equation, which we have found to decay exponentially as $t \to \infty$. Therefore, the long-term behavior of our solution is dominated by the cosine term, and therefore the particular solution

$$x(t) = A\cos(\omega t - \delta)$$

is what we are usually concerned. Clearly, we can adjust this equation of motion for cases of weak/strong damping.

Definition 2.5.6 (Attractor)**.** Let us study the effects the transient terms. Clearly, the values of r_1, r_2 depend on the initial conditions x_0 and v_0, and so these initial conditions will have a big effect on the motion at first.



The transient motion is clearly visible, but after a while only the long-term motion remains, oscillating sinusoidally at *exactly the drive frequency*! Furthermore, the initial differences disappear and the motion settles down to the same sinusoidal motion of the particular solution *irrespective of the initial conditions*. Therefore, the motions corresponding to several different initial conditions are "attracted" to the particular motion, hence the name *attractor*.

2.5.6 Fourier Series

Definition 2.5.7 (τ-periodic Functions)**.** A function f is *τ-periodic* if it has period of length τ. That is,

$$f(t+\tau) = f(t) \text{ for all } t$$

The following list of function are clearly τ-periodic.

$$\cos(2\pi t/\tau),\ \cos(4\pi t/\tau),\ \cos(6\pi t/\tau), \ldots$$
$$\sin(2\pi t/\tau),\ \sin(4\pi t/\tau),\ \sin(6\pi t/\tau), \ldots$$

which can be written more compactly (setting $\omega = 2\pi/\tau$) as

$$\cos(n\omega t), \sin(n\omega t) \text{ for } n = 0, 1, 2, \ldots$$

Theorem 2.5.6 (Fourier's Theorem). Every τ-periodic function f can be expressed as the series

$$f(t) = \sum_{n=0}^{\infty} \big(a_n \cos(n\omega t) + b_n \sin(n\omega t)\big)$$

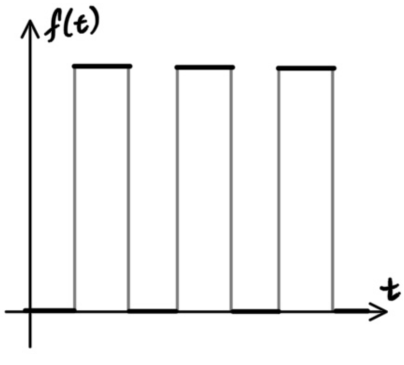
called a *Fourier series*. Furthermore, the coefficients a_n can be computed with the formulas

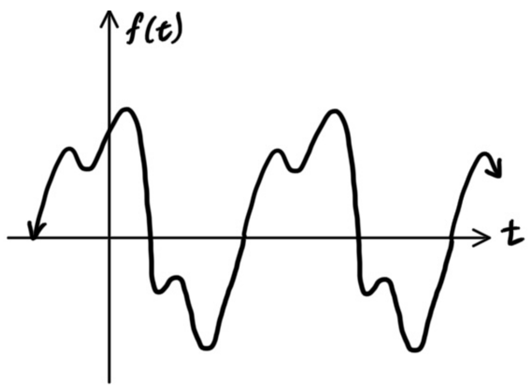
$$a_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\,dt, \quad a_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\,\cos(n\omega t)\,dt \text{ for } n \geq 1$$

along with the b_n's being

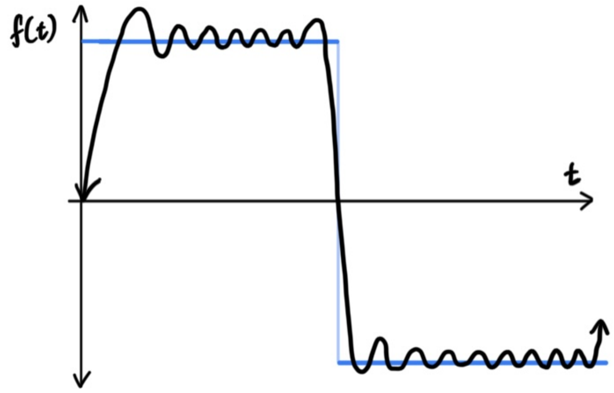
$$b_0 = 0, \quad b_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\,\sin(n\omega t)\,dt \text{ for } n \geq 1$$

This is quite surprising, since it means that the the arbitrary functions, which don't even need to be continuous (like the one on the left), can be represented as the sum of smooth ones.





It also presents an excellent approximation by retaining the first few terms.



Definition 2.5.8 (Fourier Series Solution of Driven Oscillator)**.** Given the differential equation of motion

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f$$

let the driving force f be written in its Fourier series form

$$f(t) = \sum_{n=0}^{\infty} f_n \cos(n\omega t)$$

Then, the Fourier series solution to the differential equation is

$$x(t) = \sum_{n=0}^{\infty} A_n \cos(n\omega t - \delta_n)$$

where

$$A_n = \frac{f_n}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + 4\beta^2 n^2 \omega^2}}, \quad \delta_n = \arctan\left(\frac{2\beta n\omega}{\omega_0^2 - n^2\omega^2}\right)$$

To summarize, the steps to finding the Fourier series solution to the linear inhomogeneous differential equation is to:

1. Find the coefficients f_n of the Fourier series for the given driving force $f(t)$
2. Calculate the quantities A_n and δ_n with the formulas above.
3. Write down the solution $x(t)$ as the Fourier series.

2.6 Calculus of Variations

2.7 Lagrangian Mechanics

2.7.1 Constraints

We may infer that all problems in classial mechanics have been reduced to solving the system of differential equations

$$m_i\ddot{r}_i = F_i^{(e)} - \sum_j F_{ji}$$

by substituting the known forces acting on the particle. However, we must take a look at the *constraints* that limit the motion of the system. Some examples of constraints are:

1. Dealing with rigid bodies, which is a system of particles that can only be moved through isometries.
2. Gas molecules constrained within a container are constrained by the walls of the container.
3. A particle allowed to move only within a certain surface or manifold.

Definition 2.7.1. If the constraints can be expressed as equations connecting the coordinates of the particles (and possibly the time) having the form

$$f\left(r_1, r_2, ..., t\right) = 0 \tag{2.3}$$

then the constraints are said to be *holonomic.*

Example 2.7.1. *The constraints of rigid bodies are holonomic since it can be expressed with the equations*

$$\left(r_i - r_j\right)^2 - c_{ij}^2 = 0$$

Example 2.7.2. *A particle constrained to move along a surface is a holonomic constraint, defined by the equation of the surface.*

Example 2.7.3. *The walls of a gas container is a nonholonomic constraint.*

Example 2.7.4. *The constraint describing the motion of a particle placed on a sphere is nonholonomic, since it can be described with the inequality*

$$r^2 - a^2 \geq 0$$

Definition 2.7.2. Constraint equations that contain time as an explicit variable (meaning that the physical constraints are changing) are called *rheonomous.* Ones that are not explicitly dependent on time are called *scleronomous.*

Note that if the constraint moes as a reaction to the particle's motion, then the time paramater is really dependent on the components of the particle's radius vector. This means that the system is *scleronomous.*

$$f\left(r_1, r_2, ..., t\right) = f\left(r_1, r_2, ..., t(r_1, r_2, ...)\right)$$

For more complicated systems, it is necessary to step away from Cartesian coordinates and use a basis transformation to convert them to *generalized coordinates* with a certain degree of freedom.

A system of N particles in three dimensional space has $3N$ independent coordinates, which translates to $3N$ degrees of freedom. Given k holonomic constraints given in the form (1), the system has $3N - k$ degrees of freedom. A convenient way to integrate these constraint equations is with transformation equations. With $3N - k$ degrees of freedom, we define the new set of independent variables $q_1, q_2, ..., q_{3N-k}$. This induces a transformation T from the q-variables to the r-variables.

$$T : (q_1, q_2, ..., q_{3N-k}) \mapsto (r_1(q), r_2(q), ..., r_{3N}(q))$$

where $q = (q_1, q_2, ..., q_{3N-k})$. Equivalently, we write the radius vectors of each particle in terms of the new coordinates, with possibly a time variable t. This gives the set of N *transformation equations*

$$\begin{aligned} r_1 &= r_1(q_1, q_2, ..., q_{3N-k}, t) \\ r_2 &= r_2(q_1, q_2, ..., q_{3N-k}, t) \\ ... &= ... \\ r_N &= r_N(q_1, q_2, ..., q_{3N-k}, t) \end{aligned}$$

which constraints the motion of the particles implicitly. It is assumed that we can always invert the transformation. That is, let $Q \subset \mathbb{R}^3$ be the constrained space. Given $T : Q \longrightarrow \mathbb{R}^3$, the restriction of T^{-1} onto the image of $T$is well-defined and surjective

$$T^{-1} : \text{Im}(T) \subset \mathbb{R}^3 \longrightarrow Q$$

Furthermore, note that T is a homeomorphism. This leads to a very important realization that Q is $(3N - k)$-manifold embedded in the $3N$-manifold $\mathbb{R}$. This set Q consisting of points that represent viable configurations of the system is called the *configuration manifold* of the system. The existence of the chart mappings guarantees that every point is locally paramaterizable with new coordinates (which, in this case, is the q-variables).

Example 2.7.5. *In the case that a particle is constrained to move on the surface of a sphere, two angles θ, ϕ representing latitude and longitude are obvious possible generalized coordinates.*

Example 2.7.6. *A double pendulum moving in a plane can be represented by two angles.*

Example 2.7.7. *A disk rolling vertically on the horizontal xy-plane.*

Holonomic constraints therefore allow us to use the tools in manifold theory to model systems of equations. However, there is no standardized method of tackling systems with nonholonomic constraints. Therefore, it is almost always assumed that constraints are holonomic, and this doesn't greatly limit the applicability of the theory.

2.7.2 D'Alembert's Principle and Lagrange's Equations

Definition 2.7.3. A system is in *equilibrium* if the total force on each particle vanishes. That is, if $F_i = 0$ for all i.

Definition 2.7.4. A *virtual (infinitesmial) displacement* of a system refers to a change in the configuration of a system as the result of any infinitesmal change of the coordinates δr_i, consistent with the forces and constraints imposed on the system at given instance t.

Note that this displacement is called virtual to distinguish it from an actual displacement of a system occurring in the time interval dt, during which the forces and constraints may be changing. To solidify this concept of a virtual displacement and its difference from the actual displacement, consider the configuration manifold Q. The time evolution of the system can be modeled with a path function $q(t)$ on Q with paramater time. The actual displacement of the system is modeled by an infinitesimal displacement on the curve.

However, we can define another path γ (a "virtual curve" as opposed to the real curve) along which the system evolves. Again, this must be consistent with the actual constraints, or equivalently, γ cannot "leave" the configuration manifold.

Now, suppose that that we are working with a system that is in equilibrium. Since the total force on each particle vanishes, then clearly the dot product of the force in direction δr_i, which is the virtual work of the particle over displacement δr_i, vanishes, too. Summing over all particles,

$$\sum_i F_i \cdot \delta r_i = 0$$

We decompose F_i into the applied force $F_i^{(a)}$ and the force of constraint f_i. Then,

$$\sum_i F_i \cdot \delta r_i = 0 \implies \sum_i F_i^{(a)} \cdot \delta r_i + \sum_i f_i \cdot \delta r_i = 0$$

Let us now restrict the system such that the net virtual work of the forces of constraint is 0.

It is preferred to simplify the mechanics of a system (e.g. paramaterizing it) in such a way that the forces of constraint disappear. It is a common problem that the forces of constraint are unknown a priori, so we try to have them vanish. Looking at the time-evolution path through the configuration manifold, we can interpret the assumption above as saying that the vectors of the forces of constraint are perpendicular to the surface of the manifold.

Example 2.7.8. *A particle constrained to move on a surface has a form of constraint perpendicular to the particle's displacement (e.g. a normal force)* $\implies$ $f_i \cdot \delta r_i = 0$*, where* δr_i *is an infinitesimal virtual displacement on the surface.*

Theorem 2.7.1 (Principle of Virtual Work). This reduces the above equation to

$$\sum_i F_i^{(a)} \cdot \delta r_i = 0$$

which states that a condition for equilibrium of a system is that the virtual work of the applied forces vanishes.

We can write this in a different way. The equation of motion can be rewritten as

$$F_i = \dot{p}_i \implies F_i - \dot{p}_i = 0$$

which states that particles in the system will be in equilibrium under a force equal to the actual force plus a "reversed effective force" $-\dot{p}_i$. This leads to

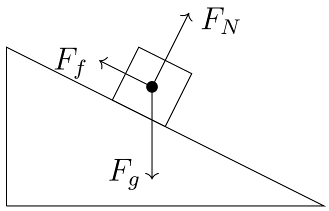
$$\sum_i \left(F_i - \dot{p}_i\right) \cdot \delta r_i = 0$$

$$\implies \sum_i \left(F_i^{(a)} - \dot{p}_i \cdot \delta r_i + \sum_i f_i \cdot \delta r_i = 0\right.$$

where $F_i^{(a)}$ represents the applied forces and f_i represents the constraint forces. We again restrict our work to systems where the virtual work of the forces of constraints vanishes, to get

$$\sum_i \left(F_i^{(a)} - \dot{p}_i\right) \cdot \delta r_i = 0$$

Example 2.7.9. *In the diagram, we can see that the normal force F_N is the constraint force, while friction F_f and gravity F_g are applied forces.*



Furthermore, this system has the property that $f_i \cdot \delta r_i = 0$, since the normal force is always perpendicular to the back-and-forth movement of the block. Note that if the box leaves the surface and into the air (which is viable, since the system isn't constrained by the ramp and ground), the normal force then vanishes and is still consistent with the system.

If the system is in equilibrium, then

$$F = F_f + F_N + F_g = 0$$

and the virtual displacement from, say point A to point B, denoted δr, cancels out with the total force F (since $F = 0$). That is,

$$F \cdot \delta r = 0 \cdot \delta r = 0$$

Therefore, the virtual displacement of a system in equilibrium is δr with component displacement δr_i and the virtual work always 0.

Definition 2.7.5. Let $\mathcal{M}$ be the configuration manifold of a mechanical system, with $t_0, t_1 \in \mathbb{R}$ time constants, $q_0, q_1 \in \mathcal{M}$, and

$$P(\mathcal{M}) \equiv \left\{\gamma \in C^\infty([t_0, t_1], \mathcal{M}) \mid \gamma(t_0) = q_0, \gamma(t_1) = q_1\right\}$$

For each path $\gamma \in P(\mathcal{M})$ and $\varepsilon_0 > 0$, a *variation* of γ is a function

$$\Gamma : [t_0, t_1] \times [-\varepsilon_0, \varepsilon] \longrightarrow \mathcal{M}$$

such that for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$,

$$\Gamma(\cdot, \epsilon) \in P(\mathcal{M}) \text{ and } \Gamma(t, 0) = \gamma(t)$$

Alternatively, we can imagine the variation Γ as an infinitesimal homotopy of γ in $\mathcal{M}$.

Going back to D'Alembert's principle, we let $F_i = F_i^{(a)}$ and have

$$\sum_i \left(F_i - \dot{p}_i\right) \cdot \delta r_i = 0$$

Since

$$r_i = r_i(q_1, q_2, ..., q_n, t)$$

with generalized coordinates q_i (assuming independent coordinates), we use the multivariate chain rule to get

$$V_i \equiv \frac{dr_i}{dt} = \left(\sum_k \frac{\partial r_i}{\partial q_k}\dot{q}_k\right) + \frac{\partial r_i}{\partial t}$$

Analogously, we find the displacement

$$\delta r_i = \sum_j \frac{\partial r_i}{\partial q_j}\delta q_j$$

Note that no variation of time, δt, is involved here since virtual displacement by definition considers displacements of the coordinates. Therefore, in generalized coordinates, the virtual work of F_i is

$$\begin{aligned}\sum_i F_i \cdot \delta r_i &= \sum_{i,j} F_i \cdot \frac{\partial f_i}{\partial q_j}\delta q_j \\ &= \sum_j Q_j \delta q_j, \text{ where } Q_j = \sum_i F_i \frac{\partial r_i}{\partial q_j}\end{aligned}$$

Note that even though $Q_j \delta q_j$ must always have the dimensions of work, the Q's do not necessarily need to have the dimensions of force since the q's need not have dimensions of length.

Example 2.7.10. *Q_j may be a value of torque N_j and dq_j a differential angle $d\theta_j$, which makes $N_j d\theta_j$ a differential of work.*

We now evaluate

$$\sum_i \dot{p}_i \delta r_i = \sum_i m_i \ddot{r}_i \delta r_i = \sum_{i,j} m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j}\delta q_j$$

Using the product rule for derivatives, we get

$$\sum_i m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} = \sum_i \left(\frac{d}{dt}\left(m_i \dot{r}_i \frac{\partial r_i}{\partial q_j}\right) - m_i \dot{r}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right)\right)$$

By equality of partial derivatives,

$$\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) = \frac{\partial \dot{r}_i}{\partial q_j} = \left(\sum_k \frac{\partial^2 r_i}{\partial q_j \partial q_k}\dot{q}_k\right) + \frac{\partial^2 r_i}{\partial q_j \partial t} = \frac{\partial v_i}{\partial q_j}$$

Remember that we are deriving with respect to generalized q-coordinates.

Lemma 2.7.2. In this type of system,

$$\frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}$$

Proof. To be done. ■

Using the previous lemma, we get

$$\sum_i m_i\ddot{r}_i = \sum_i \left(\frac{d}{dt}\left(m_i v_i \frac{\partial v_i}{\partial \dot{q}_j}\right) - m_i v_i \frac{\partial v_i}{\partial q_j}\right)$$
$$= \sum_i \left(\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}m_i v_i^2\right)\right) - \frac{\partial}{\partial q_j}\left(\frac{1}{2}m_i v_i^2\right)\right)$$

So rewriting D'Alembert's principle, we have

$$\sum_j \left(\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2}m_i v_i^2\right)\right) - \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2}m_i v_i^2\right) - Q_j\right)\delta q_j$$

Letting $T = \sum_i \frac{1}{2}m_i v_i^2$, we get

$$\sum_j \left[\left(\frac{d}{dt}\left(\frac{\partial T}{\partial q_j}\right) - \frac{\partial T}{\partial q_j}\right) - Q_j\right]\delta q_j = 0 \tag{2.4}$$

So far, no restriction has been made in the system constraints other than they be workless in a virtual displacement. In the case that the constraint is holonomic, the generalized variables q_j will be completely independent of each other (with the constraints implicitly contained within the transformation of coordinates). This means that any virtual displacement δq_j is independent of δq_k $(j \neq k)$., so the only way that (2) can be 0 is if all the individual elements are 0 for every $j = 1, 2, ..., n$. That is,

$$\forall j = 1, 2, ..., n, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j = 0$$

If the forces can be derived from a certain scalar potential function V (i.e. F is a conservative vector field), then

$$F = -\nabla_i V \implies Q_j = \sum_i F_i \frac{\partial r_i}{\partial q_j} = -\sum_i \nabla_i V \frac{\partial r_i}{\partial q_j}$$

But this last expression is just the partial derivative of the function $-V(r_1, r_2, ..., r_n, t)$ with respect to q_j. So,

$$Q_j = -\frac{\partial V}{\partial q_j}$$

However, note that the field cannot be conservative if V is a function of time. So, V must be invariant under time for the above equations to hold.

As defined, the potential V does not depend on the generalized velocities. Hence, we can include a term V in the partial derivative with respect to $\dot{q}_j$ without changing the outcome. This leads to

$$\frac{d}{dt}\left(\frac{\partial(T-V)}{\partial \dot{q}_j}\right) - \frac{\partial(T-V)}{\partial q_j} = 0$$

Definition 2.7.6. The *Lagrangian*, denoted L, is defined

$$L = T - V$$

where T is the kinetic energy and V is the potential energy of the system.

Theorem 2.7.3 (Lagrange's Equations).

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

Note that for a particular set of equations of motion there is no unique choice of Lagrangian such that Lagrange's equations lead to the equations of motions in the given generalized coordinates. In fact, if $L(q, \dot{q}, t)$ is an appropriate Lagrangian, and $F(q, t)$ is any differentiable function of the generalized coordinates and time, then

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{\partial F}{\partial t}$$

is a Lagrangian also resulting in the same equations of motion. There are other methods to find other Lagrangians, too.

Chapter 3

Electromagnetism

Chapter 4

General Relativity

Chapter 5

Quantum Mechanics

Chapter 6

Quantum Field Theory

Chapter 7

String Theory

Appendix A

Further Readings

1. Herbert Goldstein. *Classical Mechanics*