Complex Analysis

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Contents

The motivation for complex numbers is quite established as the field extension $\mathbb{R} \subset \mathbb{R}[i] \simeq \mathbb{C}$ which gives algebraic closure.

1 The Complex Numbers

The next field that will be particularly important is the complex numbers. It is straightforward to construct $\mathbb{C}$, but let's motivate this for a minute.

Example 1.1 (Polynomial Roots)

The roots of the polynomial

$$f(x) = x^2 + 1 \tag{1}$$

does not exist in $\mathbb{R}$.

Therefore, we would like to construct a new space that contains all possible roots for all possible polynomials with real coefficients. We call this $\mathbb{C}$. Clearly, by constructing polynomials of the form $x^2 - r^2$ for some $r \in \mathbb{R}$, we know that $\mathbb{R} \subset \mathbb{C}$. Therefore, we want to create a further extension of $\mathbb{R}$, along with some canonical injection $\iota : \mathbb{R} \to \mathbb{C}$ that is also a field homomorphism. It turns out that once we construct this field, there is no possible way that we can make it an ordered field. However, the norm extends naturally into $\mathbb{C}$ such that ι is isometric. Finally, we can define a new operator called *conjugation* that gives us additional structure.

This is not the only way to construct the complex plane however. Rather than defining all these from scratch, we could just define the addition operations with an isometric vector space isomorphism from $\mathbb{R}^2$ to $\mathbb{C}$ actually, and then define multiplication. Another way is to start again with $\mathbb{Q} \times \mathbb{Q}$, define a norm on it, complete it, and finally define the addition and multiplication operations that satisfy the field property.

1.1 Construction

Theorem 1.1 (Construction of the Complex Numbers)

Let $\mathbb{C}$ be defined as the space $\mathbb{R} \times \mathbb{R}$ with the following operations.

1. *Addition.* $x = (a, b), y = (c, d) \implies x +_{\mathbb{C}} y = (a + c, b + d)$.
2. *Additive Identity.* $0_{\mathbb{C}} = (0, 0)$.
3. *Additive Inverse.* $x = (a, b) \implies -x = (-a, -b)$.
4. *Multiplication.* $x = (a, b), y = (c, d) \implies x \times_{\mathbb{C}} y = (ac - bd, ad + bc)$.
5. *Multiplicative Identity.* $1_{\mathbb{C}} = (1, 0)$.
6. *Multiplicative Inverse.*

$$x = (a, b) \implies x^{-1} = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right) \tag{2}$$

Our first claim is that $(\mathbb{C}, +_{\mathbb{C}}, \times_{\mathbb{C}})$ is a field. Furthermore, we define the additional structures

1. *Conjugate.* $x = (a, b) \implies \overline{x} = (a, -b)$.
2. *Norm.* $|x|_{\mathbb{C}} = x \times_{\mathbb{C}} \overline{x} = a^2 + b^2$.
3. *Metric.* This is the norm-induced metric. $d_{\mathbb{C}}(x, y) = |x - y|_{\mathbb{C}}$.
4. *Topology.* This is the metric-induced topology generated by the open balls $B(x, r) \coloneqq \{y \in \mathbb{C} | d(x, y) < r\}$, where $x \in \mathbb{C}, r \in \mathbb{R}$.

Our second claim is that the canonical injection $\iota : \mathbb{R} \to \mathbb{C}$ defined

$$\iota(r) = (r, 0) \tag{3}$$

is an isometric field isomorphism. Our third claim is that $\mathbb{C}$ is Cauchy-complete with respect to this metric.

Note that we do not talk about order $\mathbb{C}$, and so the concepts of Dedekind completeness, least upper bound properties, or Archimedean principle is meaningless in the complex plane.

Definition 1.1 (Imaginary Number)

Let us denote $i = (0, 1)$ which we call the **imaginary number**, which has the property that $i^2 = 1$. With this notation, we can see through abuse of notation that

$$(a, b) = (a, 0) + (0, b) = (a, 0) + (b, 0)(0, 1) = a + bi \tag{4}$$

Therefore, we generally write complex numbers as $z = a + bi$, and we define the real and imaginary components as $\text{Re}(z)$ and $\text{Im}(z)$, respectively.

Note that the identity $x^2 + 1 \equiv (x + i)(x - i)$ implies that the equation $x^2 = -1$ has exactly two solutions in $\mathbb{C}$, i and $-i$. Therefore, if a subfield of $\mathbb{C}$ contains one of these solutions, it must contain the other (since i and $-i$ are additive and multiplicative inverses).

Furthermore, since i is defined to be $\sqrt{-1}$, we could replace i with $-i$ and our calculations would still be consistent throughout the rest of mathematics. In fact, i and $-i$ behave **exactly** identically and cannot be distinguished in an abstract sense. Visually, the complex plane "flipped" across the real number axis produces the same complex plane.

Theorem 1.2 (Uniqueness of $\mathbb{C}$)

$\mathbb{C}$ is unique up to an isomorphism that maps all real numbers to themselves. Every complex number can be uniquely written as $a + bi$, where $a, b \in \mathbb{R}$ and i is a fixed element such that $i^2 = -1$.

Proof.

Consider the subset of $\mathbb{C}$

$$K \equiv \{a + bi \mid a, b \in \mathbb{R}\} \tag{5}$$

By evaluating its operations, we can check for closure, identity, and invertibility of nonzero elements to conclude that K is a subfield of $\mathbb{C} \implies$ by prop. (iii), $K = \mathbb{C} \implies$ every element in $\mathbb{C}$ can be written in form $a + bi$. To prove uniqueness, we assume that $p \in \mathbb{C}$ can be written in distinct forms $p = a + bi = a' + b'i$. Then

$$\begin{aligned} a + bi = a' + b'i &\implies (a - a')^2 = (b'i - bi)^2 = -(b' - b)^2 \\ &\implies a - a' = b' - b = 0 \end{aligned}$$

To prove uniqueness of $\mathbb{C}$ up to ismorphism, we assume that $\mathbb{C}'$ exists with i' such that i'^2 containing elements $a + bi'$. Let $f : \mathbb{C} \longrightarrow \mathbb{C}'$ defined

$$f(a + bi) = a + bi' \tag{6}$$

Then,

$$\begin{aligned} f\big((a + bi) + (c + di)\big) &= f\big((a + c) + (b + d)i\big) \\ &= (a + c) + (b + d)i' \\ &= (a + bi') + (c + di') \\ &= f(a + bi) + f(c + di) \\ f\big(\kappa(a + bi)\big) &= f\big(\kappa a + \kappa bi\big) \\ &= \kappa a + \kappa bi' \\ &= \kappa(a + bi') \\ &= \kappa f(a + bi) \end{aligned}$$

So, f is an isomorphism, and $\mathbb{C} \simeq \mathbb{C}'$. From analysis, we can construct and prove the existence of $\mathbb{R}$. We then define the map

$$\rho : \mathbb{R}^2 \longrightarrow \mathbb{C},\ \rho(a, b) \equiv a + bi \tag{7}$$

with $\rho(1,0)$ as the multiplicative identity and $\rho(0,1) \equiv i$. Therefore, every element of $\mathbb{C}$ can be uniquely represented as an element of $\mathbb{R}^2$.

Unfortunately, we lose the ordering.

Theorem 1.3 (Order on Complex Plane)

There exists no order on $\mathbb{C}$ that makes it a totally ordered field.

Proof.

We attempt to construct an order on i and 0 in $\mathbb{C}$.

1. If $i = 0$, then $i^4 = 0 \cdot i^3 \implies 1 = 0$, which contradicts that $0 < 1$.
2. If $i \neq 0$, then $i^2 > 0$ from the field axioms, and so $-1 > 0$. But this also means that $1 = i^4 > 0$. This contradicts the ordered field property that $x > 0 \iff -x < 0$.

Therefore $\mathbb{C}$ cannot be turned into an ordered field.

1.2 Properties of the Complex Plane

Theorem 1.4 (Conjugation is an Isomorphism)

Conjugation is an isometric field automorphism of $\mathbb{C}$.

$$c = a + bi \mapsto \bar{c} = a - bi \tag{8}$$

This is identically defined by replacing i with $-i$. Clearly, $\bar{\bar{c}} = c$.

Proof.

Proposition 1.1 (Properties of Conjugation)

For any $c \in \mathbb{C}$, $c + \bar{c}$ and $c\bar{c}$ are real.

Proof.

Using the fact that the complex conjugate is an isomorphism,

$$\begin{aligned} &\overline{c + \bar{c}} = \bar{c} + \bar{\bar{c}} = \bar{c} + c = c + \bar{c} \\ &\overline{c\bar{c}} = \bar{c}\bar{\bar{c}} = \bar{c}c = c\bar{c} \end{aligned}$$

Note that we proved this abstractly using only the properties given above, and did not decompose c to its **algebraic form** $a + bi$.

If $c = a + bi,\ a, b \in \mathbb{R}$, then

$$c + \bar{c} = 2a,\ c\bar{c} = a^2 + b^2 \tag{9}$$

1.3 Polar Coordinates

In case the reader is unaware, it is common to interpret complex numbers $c = a + bi$ as points or vectors (a, b) on the complex plane.

Definition 1.2 (Polar Form of Complex Numbers)

The **polar representation**, or **trigonometric representation**, of a complex number $c = a + bi$ is defined using the equations

$$a = r\cos\varphi,\ b = r\sin\varphi \implies c = r(\cos\varphi + i\sin\varphi) \tag{10}$$

where $r = |c|$ and φ is the **argument** of c, which is the angle formed by the corresponding vector with the polar axis defined within the interval $[0, 2\pi)$.

$$\arg(c) \equiv \tan^{-1}\frac{b}{a} \tag{11}$$

This mapping can be defined

$$\rho : \mathbb{R} \times \frac{\mathbb{R}}{2\pi} \longrightarrow \mathbb{C},\ \rho(r,\varphi) = r(\cos\varphi + i\sin\varphi) \tag{12}$$

Theorem 1.5 ()

ρ is "similar" to a homomorphism in the following way. By defining the domain and codomain as groups,

$$\rho : \big(\mathbb{R}, \times\big) \times \bigg(\frac{\mathbb{R}}{2\pi}\bigg) \longrightarrow \big(\mathbb{C}, \times\big) \tag{13}$$

we can see that

$$\rho(r_1, \varphi_1) \times \rho(r_2, \varphi_2) = \rho(r_1 \times r_2, \varphi_1 + \varphi_2) \tag{14}$$

or equivalently,

$$r_1(\cos\varphi_1 + i\sin\varphi_1) \cdot r_2(\cos\varphi_2 + i\sin\varphi_2) = r_1 r_2(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)) \tag{15}$$

Corollary 1.1 ()

The formula for the ratio of complex numbers is defined

$$\frac{r_1(\cos\varphi_1 + i\sin\varphi_1)}{r_2(\cos\varphi_2 + i\sin\varphi_2)} = \frac{r_1}{r_2}(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)) \tag{16}$$

Corollary 1.2 ()

The positive integer power of a complex number can be written using **De Moivre's formula**.

$$\big(r(\cos\varphi + i\sin\varphi)\big)^n = r^n(\cos n\varphi + i\sin n\varphi) \tag{17}$$

1.4 Roots, Exponentials, Logarithms

We can use this formula to extract a root of nth degree from a complex number $c = r(\cos\varphi + i\sin\varphi)$, which means to solve the equation $z^n = c$. Let $z = s(\cos\psi + i\sin\psi)$. Then by De Moivre's formula,

$$\begin{aligned}
z^n &= s^n(\cos n\psi + i\sin n\psi) = r(\cos\varphi + i\sin\varphi) \\
&\implies s = \sqrt[n]{r},\ \psi = \frac{\varphi + 2\pi k}{n} \\
&\implies z = \sqrt[n]{r}\bigg(\cos\frac{\varphi + 2\pi k}{n} + i\sin\frac{\varphi + 2\pi k}{n}\bigg) \text{ for } k = 0, 1, ..., n-1
\end{aligned}$$

Geometrically, the n solutions lie at the vertices of a regular n-gon centered at the origin. When $c = 1$, the solutions are the nth roots of unity.

1.5 Trigonometric Functions

Now with complex numbers, we have a yet another way of defining trigonometric functions that generalizes that of the reals. We can use the series representation.

1.6 Dual Numbers

Another similar number system.

2 TBD

2.1 Algebraic Extension of Field $\mathbb{R}$

We introduce the number i, called the **imaginary unit**, such that $i^2 = -1$. We may multiply real numbers y to i to get yi, and we can add real numbers to such numbers, to get numbers of the form

$$x + yi, \quad x, y \in \mathbb{R}$$

We then define all objects of the form $x + iy$ as the **complex numbers**, with addition defined

$$(x_1 + iy_1) + (x_2 + iy_2) \equiv (x_1 + x_2) + i(y_1 + y_2)$$

and multiplication defined

$$(x_1 + iy_1) \cdot (x_2 + iy_2) \equiv (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

As expected, this makes $+$ and $\cdot$ commutative operations. Furthermore, two complex numbers $z = x_1 + iy_1$ and $w = x_2 + iy_2$ are equal if and only if $x_1 = x_2$ and $y_1 = y_2$.

One nontrivial property of field $\mathbb{C}$ is that every element $z \in \mathbb{C}$ has a multiplicative inverse z^{-1}. To find this, we must define the following.

Definition 2.1 (Complex Conjugate)

Given complex number $z = x + iy$, its **complex conjugate** is

$$\overline{z} = \overline{x + iy} = x - iy$$

Note that

$$z \cdot \overline{z} = x^2 + y^2 \neq 0 \text{ iff } z \neq 0$$

Thus, given z,

$$z^{-1} = \frac{1}{z \cdot \overline{z}} \cdot \overline{z} \iff (x + yi)^{-1} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

2.1.1 Geometric Interpretation of $\mathbb{C}$

Once the algebraic operations $+$ and $\cdot$ has been introduced, the symbol i is no longer needed. That is, we can define a new set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ with the operations $+_{\mathbb{R}}, \cdot_{\mathbb{R}} : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined

$$\begin{aligned} (x_1, y_1) +_{\mathbb{R}} (x_2, y_2) &\equiv (x_1 + x_2, y_1 + y_2) \\ (x_1, y_1) \cdot_{\mathbb{R}} (x_2, y_2) &\equiv (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1) \end{aligned}$$

We can check that this new set $(\mathbb{R}^2, +_{\mathbb{R}}, \cdot_{\mathbb{R}})$ is isomorphic to $(\mathbb{C}, +, \cdot)$ as fields, and therefore one can identify complex numbers with vectors $z = (x, y)$ of the plane $\mathbb{R}^2$, where $x = \operatorname{Re} z$ is called the **real part** and $y = \operatorname{Im} z$ is called the **imaginary part**.

Definition 2.2 (Norm, Metric of $\mathbb{C}$)

Moreover, the isomorphism

$$\gamma : \mathbb{C} \longrightarrow \mathbb{R}^2, \;\; \gamma(x + yi) = (x, y)$$

induces additional structures on $\mathbb{C}$, such as the norm and metric.

1. The norm of $z = x + iy \in \mathbb{C}$ is defined as the norm of $\gamma(z) = (x, y) \in \mathbb{R}^2$. That is,

$$|z| = |x + yi| = |(x, y)| = \sqrt{x^2 + y^2}$$

Or more simply,

$$|z| = z \cdot \overline{z}$$

2. The metric of two complex numbers $z_1, z_2 \in \mathbb{C}$ is defined

$$|z_1 - z_2| = |(x_1, y_1) - (x_2, y_2)| = |(x_1 - x_2, y_1 - y_2)| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Or more simply,

$$|z_1 - z_2| = (z_1 - z_2) \cdot \overline{(z_1 - z_2)}$$

Definition 2.3 (Polar Coordinates of $\mathbb{C}$)

Given the basis transformation of polar coordinates $(r, \varphi) \mapsto p(r, \varphi) = (x, y)$ where

$$p \begin{pmatrix} r \\ \varphi \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

the isomorphism $\mathbb{C} \simeq \mathbb{R}^2$ induces a similar polar transformation in $\mathbb{C}$

$$\rho = \gamma^{-1} \circ p \circ \gamma : \mathbb{C}_{(r,\theta)} \longrightarrow \mathbb{C}_{(x,y)}, \quad \rho(r + \theta i) = r\cos\theta + r\sin\theta i = x + yi$$

as shown in the commutative diagram.

$$\begin{tikzcd} \mathbb{C}_{(r,\theta)} \arrow[r, "\rho"] \arrow[d, "\gamma"] & \mathbb{C}_{(x,y)} \arrow[d, "\gamma"] \\ \mathbb{R}^2_{(r,\theta)} \arrow[r, "p"] & \mathbb{R}^2_{(x,y)} \end{tikzcd}$$

Therefore, we can write

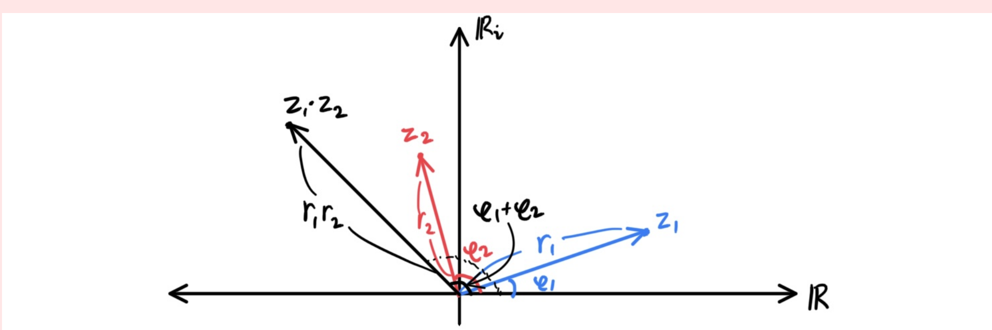
$$z = r(\cos\varphi + i\sin\varphi)$$

where $r = |z|$ is called the **magnitude** of z, and $\varphi = \operatorname{Arg} z$ is called the **argument** of z.

Lemma 2.1 (Multiplication of Complex Numbers in Polar Form)

It turns out that multiplication is a lot easier in polar coordinates than in rectangular ones:

$$\begin{aligned} z_1 \cdot z_2 &= (r_1\cos\varphi_1 + ir_1\sin\varphi_1)(r_2\cos\varphi_2 + ir_2\sin\varphi_2) \\ &= \ldots \\ &= r_1 r_2 \big(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)\big) \end{aligned}$$



Theorem 2.1 (De Moivre's Formula)

By induction using the previous lemma, we get

$$z = r(\cos\varphi + i\sin\varphi) \implies z^n = r^n(\cos n\varphi + i\sin n\varphi)$$

Corollary 2.1 (Roots of Unity)

The n complex solutions of the equation

$$z^n = a$$

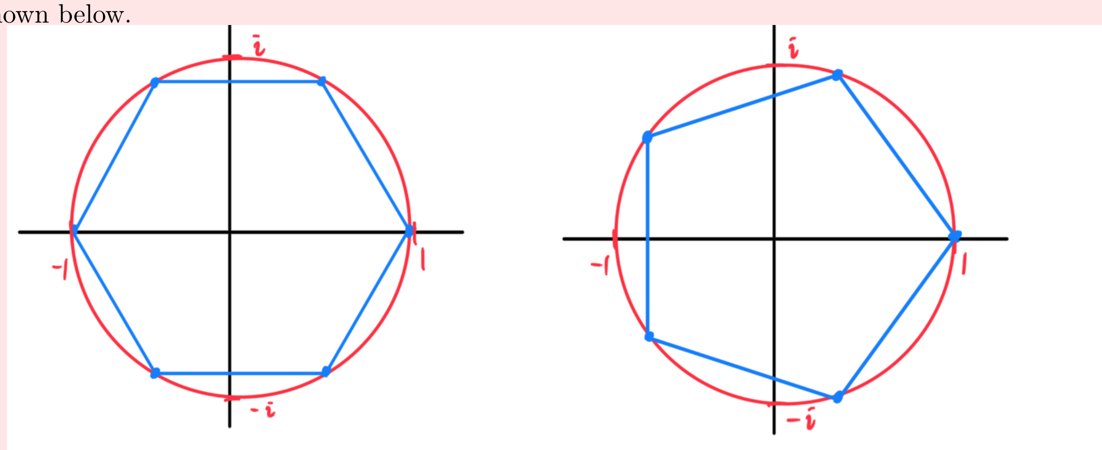
where $a = \rho(\cos\psi + i\sin\psi)$ is

$$z_k = \sqrt[n]{\rho}\bigg(\cos\Big(\frac{\psi + 2\pi k}{n}\Big) + i\sin\Big(\frac{\psi + 2\pi k}{n}\Big)\bigg), \qquad k = 0, 1, 2, \ldots, n-1$$

Moreover, if $a = 1$, then the n complex solutions are called the n**th roots of unity**, defined

$$z_k = \cos\Big(\frac{2\pi k}{n}\Big) + i\sin\Big(\frac{2\pi k}{n}\Big), \qquad k = 0, 1, 2, \ldots, n-1$$

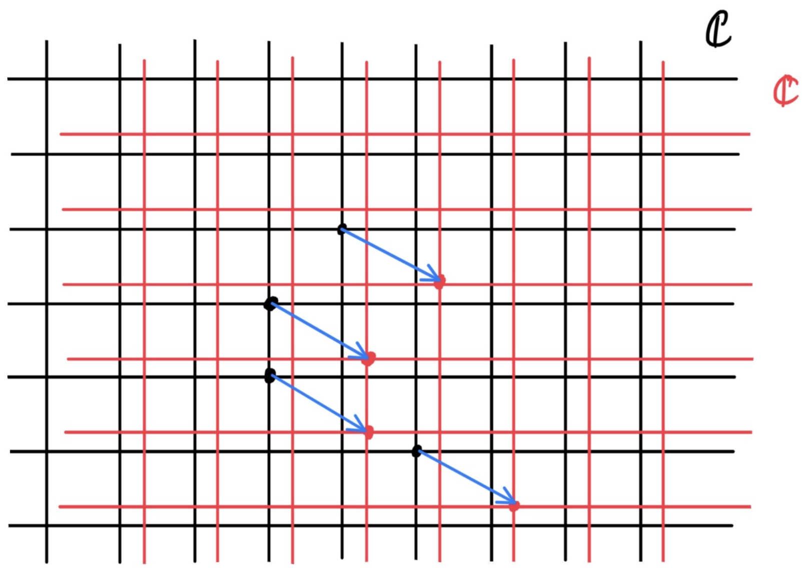
which shows that the nth roots of unity are at the vertices of a regular n-sided polygon inscribed in the unit circle, with one vertex at 1, within the complex plane. The 5th and 6th roots of unity are shown below.



Finally, we can visualize certain transformations in $\mathbb{C}$. For a fixed $b \in \mathbb{C}$, the sum $z + b$ cam be interpreted as the mapping of $\mathbb{C}$ onto itself given by the formula

$$z \mapsto z + b$$

This mapping is a translation of the plane by the vector b.



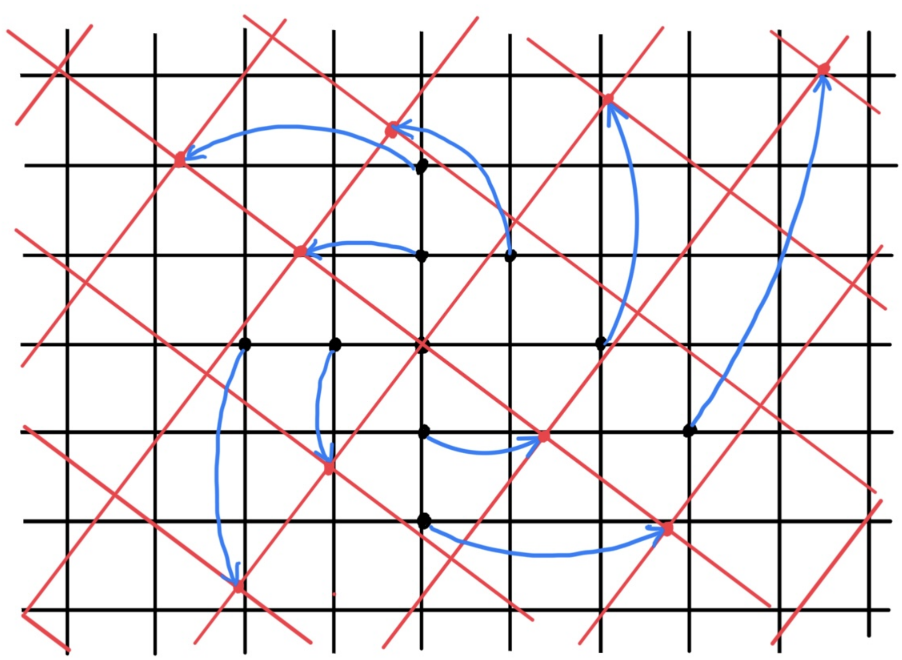
Visualizing multiplication is a bit harder. Given a

$$a = |a|(\cos\varphi + i\sin\varphi) \neq 0$$

the product az can be interpreted as the mapping of $\mathbb{C}$ onto itself given by the formula

$$z \mapsto az$$

which is the composition of a dilation by a factor of $|a|$ and a rotation through the angle $\varphi \in \operatorname{Arg} a$.

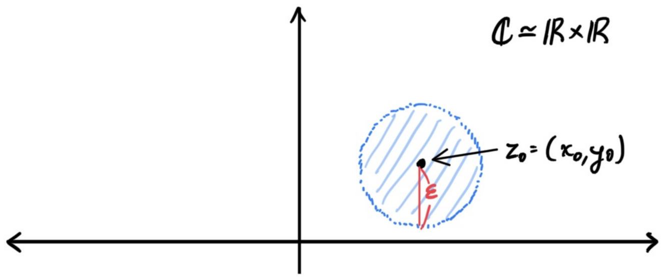


2.2 Sequences and Series in $\mathbb{C}$

Our previous construction of a metric within $\mathbb{C}$ enables to define the ϵ-neighborhood of a number $z_0 \in \mathbb{C}$ as the set

$$U_\epsilon(z_0) \equiv \{z \in \mathbb{C} \mid |z - z_0| < \epsilon\}$$

which can be visualized as an open disk of radius ϵ in $\mathbb{R}^2$ centered at point (x_0, y_0) if $z_0 = x_0 + iy_0$.



Definition 2.4 (Convergence of a Sequence in $\mathbb{C}$)

A sequence $\{z_n\}$ of complex numbers **converges** to $z_0 \in \mathbb{C}$ if and only if

$$\lim_{n\to\infty} |z_n - z_0| = 0$$

It is clear from the inequality

$$\max\{|x_n - x_0|, |y_n - y_0|\} \leq |z_n - z_0| \leq |x_n - x_0| + |y_n - y_0|$$

that a sequence of complex numbers converges if and only if the sequences of its real and imaginary parts of the terms of the sequence both converge. That is,

$$\{z_n\} \text{ converges} \iff \{\operatorname{Re} z\} \text{ and } \{\operatorname{Im} z\} \text{ converges}$$

Lemma 2.2 (Convergence of Cauchy Sequences over $\mathbb{C}$)

A sequence of complex numbers $\{z_n\}$ is called a **Cauchy sequence** if for every $\epsilon > 0$ there exists an index $N \in \mathbb{N}$ such that

$$|z_n - z_m| < \epsilon \text{ for all } n, m > N$$

It is also clear that

$$\{z_n\} \text{ is Cauchy} \iff \{\operatorname{Re} z\} \text{ and } \{\operatorname{Im} z\} \text{ is Cauchy}$$

and using the Cauchy criterion for sequences of real numbers, we can easily see that a sequence of complex numbers converges if and only if it is a Cauchy sequence.

Lemma 2.3 (Convergence of Cauchy Series over $\mathbb{C}$)

Interpreting the sum of a series of complex numbers

$$z_1 + z_2 + \ldots + z_n + \ldots$$

as the limit of the sequence its partial sums $\{s_n\}$, where $s_n = z_1 + \ldots z_n$ as $n \to \infty$, we can see that the series converges if and only if for every $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that

$$|z_m + \ldots + z_n| < \epsilon$$

for any natural numbers $n \geq m > N$.

Definition 2.5 (Absolute Convergence of $\mathbb{C}$)

A series $z_1 + \ldots + z_n + \ldots$ of complex numbers is **absolutely convergent** if the series

$$|z_1| + |z_2| + \ldots + |z_n| + \ldots$$

converges. Clearly, is a series converges absolutely, then it converges due to the inequality

$$|z_m + \ldots + z_n| \leq |z_m| + \ldots + |z_n|$$

Example 2.1 ()

The following complex series converges because they converges absolutely. That is,

$$\begin{aligned} 1 + \frac{1}{1!}|z| + \frac{1}{2!}|z^2| + \ldots \text{ converges } \forall\, \mathbb{C} &\implies 1 + \frac{1}{1!}z + \frac{1}{2!}z^2 + \ldots \text{ converges } \forall\, \mathbb{C} \\ |z| + \frac{1}{3!}|z|^3 + \frac{1}{5!}|z|^5 + \ldots \text{ converges } \forall\, \mathbb{C} &\implies z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \ldots \text{ converges } \forall\, \mathbb{C} \\ 1 + \frac{1}{2!}|z|^2 + \frac{1}{4!}|z|^4 + \ldots \text{ converges } \forall\, \mathbb{C} &\implies 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \ldots \text{ converges } \forall\, \mathbb{C} \end{aligned}$$

Definition 2.6 (Complex Power Series)

Series of the form

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n = c_0 + c_1(z - z_0) + \ldots + c_n(z - z_0) + \ldots$$

are called **complex power series**, or **power series over** $\mathbb{C}$.

But a power series is quite useless unless we know the domain in which is converges (again, note that it is not always guaranteed to converge onto the function f if its power series expansion does converge at all). To develop more sophisticated tests of convergence of a complex power series, we introduce the complex analogue of the root test for real power series.

Theorem 2.2 (Cauchy-Hadamard Theorem)

The complex power series

$$c_0 + c_1(z - z_0) + \ldots + c_n(z - z_0) + \ldots$$

converges inside the disk $|z - z_0| < R$ with center at z_0 and radius given by the formula

$$R = \frac{1}{\overline{\lim}_{n\to\infty} \sqrt[n]{|c_n|}} = \frac{1}{\lim_{n\to\infty} \sup \sqrt[n]{|c_n|}}$$

Where $\overline{\lim}$ denotes the superior limit. Furthermore,

1. the power series diverges at any point exterior to the disk.
2. the power series converges absolutely at any point interior to the disk.
3. the power series is indeterminate at any point on the boundary of the disk.

Note that in the degenerate case when $R = 0$, the series converges only at the point $z = z_0$.

Corollary 2.2 (Abel's First Theorem on Power Series)

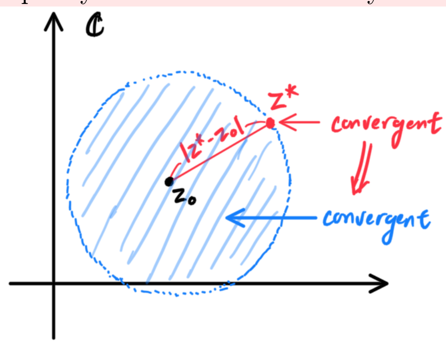
If the power series

$$c_0 + c_1(z - z_0) + \ldots + c_n(z - z_0) + \ldots$$

converges at some value z^*, then it converges absolutely for any value of z satisfying

$$|z - z_0| < |z^* - z_0|$$

The values of z satisfying the inequality above can be intuitively visualized as the following region.



Theorem 2.3 (Product of Absolutely Convergent Series)

Let $a_1 + a_2 + \ldots$ and $b_1 + b_2 + \ldots$ be an absolutely convergent series such that

$$\sum_{i=1}^{\infty} a_i = A \text{ and } \sum_{j=1}^{\infty} b_j = B$$

Then, the Cauchy product of the two series

$$\left(\sum_{i=1}^{\infty} a_i\right) \cdot \left(\sum_{j=1}^{\infty} b_j\right) = \sum_{k=0}^{\infty} c_k = AB, \text{ where } c_k = \sum_{l=0}^{k} a_l b_{k-l}$$

$a_1 b_1 + a_2 b_2 + \ldots$ is absolutely convergent and

$$\sum_{i=1}^{\infty} a_i b_i = AB$$

Proof.

To be done.

Example 2.2 (Convergence of the Cauchy Product of Absolutely Convergent Complex Series)

The two series

$$\sum_{n=0}^{\infty} \frac{1}{n!} a^n \text{ and } \sum_{m=0}^{\infty} \frac{1}{m!} b^m$$

converges absolutely. Therefore, we can see that their Cauchy product can be nicely represented by grouping together all monomials of the form $a^n b^m$ having the same total degree $m + n = k$.

$$\bigg(\sum_{n=0}^\infty \frac{1}{n!} a^n \bigg) \cdot \bigg(\sum_{m=0}^\infty \frac{1}{m!} b^m \bigg) = \sum_{k=0}^\infty \bigg(\sum_{n+m=k} \frac{1}{n!} a^n \frac{1}{m!} b^m \bigg)$$

But we can simplify

$$\sum_{m+n=k} \frac{1}{n!m!} a^n b^m = \frac{1}{k!} \sum_{n=0}^k \frac{k!}{n!(k-n)!} a^n b^{k-n} = \frac{1}{k!} (a+b)^k$$

and therefore we find that

$$\bigg(\sum_{n=0}^\infty \frac{1}{n!} a^n \bigg) \cdot \bigg(\sum_{m=0}^\infty \frac{1}{m!} b^m \bigg) = \sum_{k=0}^\infty \frac{1}{k!} (a+b)^k$$

2.3 Euler's Formula

Definition 2.7 (Complex Taylor Expansions of Transcendental Functions)

Since we have determined absolute convergence, and therefore convergence, of all these series in all of $\mathbb{C}$, it is natural to extend the definitions of

$$\exp, \cos, \sin : \mathbb{R} \longrightarrow \mathbb{R}$$

to the complex field

$$\exp, \cos, \sin : \mathbb{C} \longrightarrow \mathbb{C}$$

by defining them as

$$\begin{aligned} e^z &\equiv 1 + \frac{1}{1!} z + \frac{1}{2!} z^2 + \frac{1}{3!} z^3 + \ldots \\ \cos z &\equiv 1 - \frac{1}{2!} z^2 + \frac{1}{4!} z^4 - \frac{1}{6!} z^6 + \ldots \\ \sin z &\equiv z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 - \frac{1}{7!} z^7 + \ldots \end{aligned}$$

Notice that even in the complex field, $\cos z$ is an even function and $\sin z$ is an odd function.

$$\begin{aligned} \cos(-z) &= \cos(z) \\ \sin(-z) &= -\sin(z) \end{aligned}$$

In fact, the last example in the previous subsection just proves the following.

Lemma 2.4 (Exponential Map as a Group Homomorphism)

The exponential map $\exp : \mathbb{C} \longrightarrow \mathbb{C} \setminus \{0\}$ satisfies the following

$$\exp(z_1 + z_2) = \exp(z_1) \cdot \exp(z_2)$$

That is, exp is a group homomorphism from $(\mathbb{C}, +)$ to $(\mathbb{C} \setminus \{0\}, \cdot)$.

Definition 2.8 (Euler's Formula)

By making the substitution $z = yi$ in the series expansion of e^z (where y is an arbitrary complex number), we get

$$\begin{aligned} e^{iy} &= 1 + \frac{1}{1!}(iy) + \frac{1}{2!}(iy)^2 + \frac{1}{3!}(iy)^3 + \frac{1}{4!}(iy)^4 + \ldots \\ &= \bigg(1 - \frac{1}{2}y^2 + \frac{1}{4!}y^4 - \ldots\bigg) + i\bigg(\frac{1}{1!}y - \frac{1}{3!}y^3 + \frac{1}{5!}y^5 - \ldots\bigg) \end{aligned}$$

which brings us the identity

$$e^{iy} = \cos y + i \sin y$$

Since cos is even and sin is odd, we can add the two identities

$$\begin{aligned} e^{iz} &= \cos z + i \sin z \\ e^{-iz} &= \cos z - i \sin z \end{aligned}$$

to get

$$\begin{aligned} \cos z &= \frac{1}{2}\big(e^{iz} + e^{-iz}\big) \\ \sin z &= \frac{1}{2i}\big(e^{iz} - e^{-iz}\big) \end{aligned}$$

This gives us a very elegant connection between these three transcendental functions.

Definition 2.9 (Hyperbolic Functions)

Likewise, the following series are convergent (since they are absolutely convergent) and therefore we can define the extension of cosh and sinh into the complex field as

$$\begin{aligned} \cosh z &\equiv 1 + \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \frac{1}{6!}z^6 + \ldots \\ \sinh z &\equiv z + \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \frac{1}{7!}z^7 + \ldots \end{aligned}$$

The following identities immediately follow

$$\begin{aligned} \cosh z &= \frac{1}{2}\big(e^{z} + e^{-z}\big) \\ \sinh z &= \frac{1}{2}\big(e^{z} - e^{-z}\big) \end{aligned}$$

Lemma 2.5 (Trigonometric, Hyperbolic Identities over $\mathbb{C}$)

Common identities, which are exactly the same as their real analogues, are listed.

1. $\cos^2 z + \sin^2 z = 1$
2. $\cosh^2 z - \sinh^2 z = 1$
3. $e^{i(z_1+z_2)} = (\cos z_1 \cos z_2 - \sin z_1 \sin z_2) + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2)$
4. $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$
5. $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$
6. $\cosh z = \cos iz$
7. $\sinh z = -i \sin iz$

However, to obtain even such geometrically obvious facts as the equality

$$\sin \pi = 0 \text{ or } \cos z + 2\pi = \cos z$$

from the power series definitions of cos and sin is extremely difficult. What the properties actually do is present the remarkable unity of these seemingly different trigonometric and hyperbolic functions, which would have been impossible to detect without going into the domain of complex numbers.

If we just take the following identities

$$\begin{aligned}\cos x &= \cos(x+2\pi) \\ \sin x &= \sin(x+2\pi) \\ \cos 0 &= 1 \\ \sin 0 &= 0\end{aligned}$$

then we get the following identity.

Theorem 2.4 (Euler's Identity)

The following relation is true.

$$e^{i\pi} + 1 = 0$$

which immediately implies

$$\exp(z + 2\pi i) = \exp z$$

That is, the exponential function is a periodic function on $\mathbb{C}$ with the purely imaginary period $T = 2\pi i$.

Corollary 2.3 (Trigonometric Notation of Complex Number)

With Euler's formula and the periodic relation of $\exp z$, the trigonometric form of a complex number can be presented as

$$z = r(\cos\varphi + i\sin\varphi) = re^{i\varphi}$$

We can rewrite DeMoivre's formula as

$$z^n = r^n e^{n\varphi i}$$

2.4 Continuity, Differentiability, Analyticity of Complex Functions

The definitions of continuity and differentiability are the same, just under a different field.

Definition 2.10 (Limit of a Complex Function)

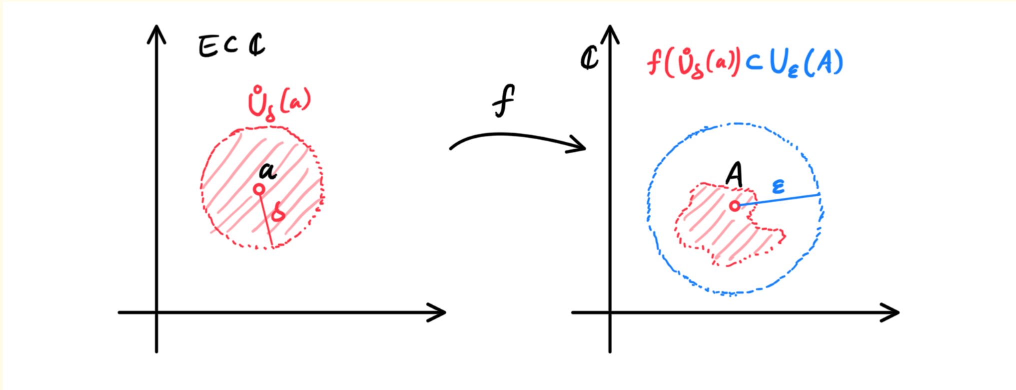
The function $f : E \subset \mathbb{C} \longrightarrow \mathbb{C}$ tends to $A \in \mathbb{C}$ as $z \to a$, or that

$$\lim_{z\to a} f(z) = A$$

if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$0 < |z - a| < \delta \implies |f(z) - A| < \epsilon$$

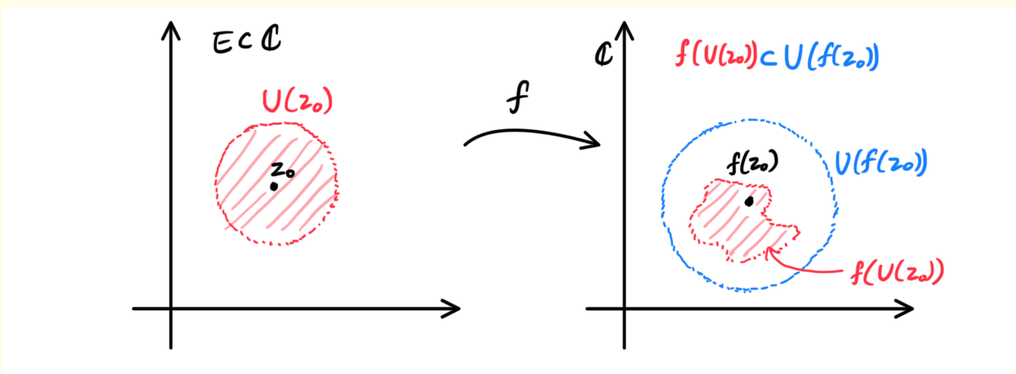
Note that we set $0 < |z - a|$ to ensure that $z \neq a$.
Therefore, in other words, for any arbitrarily small $\epsilon > 0$, we can find a $\delta > 0$ such that the image of the deleted δ-neighborhood of a, denoted $\mathring{U}_\delta(a))$, is completely within the ϵ-neighborhood $U_\epsilon(A)$.



Definition 2.11 (Continuity of a Complex Function)

A function $f : E \subset \mathbb{C} \longrightarrow \mathbb{C}$ is **continuous** at a point $z_0 \in E$ if for any neighborhood $U(f(z_0))$ there exists a neighborhood $U(z_0)$ such that its image is contained in $U(f(z_0))$. In short,

$$\lim_{z \longrightarrow z_0} f(z) = f(z_0)$$



Definition 2.12 (Differentiability of a Complex Function)

The **derivative** of a function $f : E \subset \mathbb{C} \longrightarrow \mathbb{C}$ is defined

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

if this limit exists. f **differentiable** at x_0 means that a differential function

$$df(z_0) : T_{z_0}\mathbb{C} \longrightarrow T_{f(z_0)}\mathbb{C}, \quad h \mapsto df(z_0)(h)$$

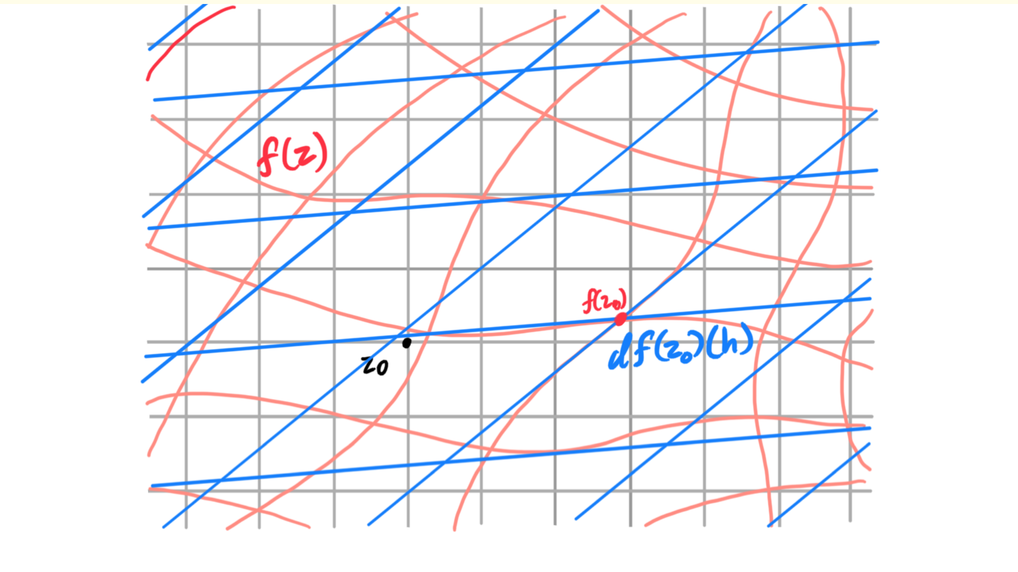
exists such that

$$f(z) = f(z_0) + df(z_0)(h) + o(h)$$

where $h = z - z_0$ is the increment of the argument. Just like the real case, it turns out that $df(z_0)(h) = f'(z_0)h$, and

$$f(z) - f(z_0) = f'(z_0)(z - z_0) + o(z - z_0)$$

which elegantly weaves together the two concepts of differentiability and the derivative.
Visualizing this, we can see that for whatever function $f : \mathbb{C} \longrightarrow \mathbb{C}$ there is a linear function that transforms the entire space as such at z_0 (along with a given point $z_0 \in \mathbb{C}$),



The differential $df(z_0)$ at the point z_0 is a linear mapping that "best" approximates f, with an error of $o(h) = o(z - z_0)$.

Lemma 2.6 (Arithmetic Properties of Differentiation over $\mathbb{C}$)

If functions $f, g : E \subset \mathbb{C} \longrightarrow \mathbb{C}$ are differentiable at a point $z \in E$, then

1. their sum is differentiable at z, and

$$d(f+g)(z) = df(z) + dg(z) \iff (f+g)'(z) = (f' + g')(z)$$

2. their product is differentiable at z, and

$$d(f \cdot g)(z) = g(z)df(z) + f(z)dg(z) \iff (f \cdot g)'(z) = f'(z)g(z) + f(z) \cdot g'(z)$$

3. their quotient is differentiable at z if $g(z) \neq 0$, and

$$d\bigg(\frac{f}{g}\bigg)(z) = \frac{g(z)df(z) - f(z)dg(z)}{g^2(z)} \iff \bigg(\frac{f}{g}\bigg)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}$$

Just like the real case, the operation of taking the derivative is a linear operator.

Lemma 2.7 (Chain Rule for Composite Functions over $\mathbb{C}$)

Let there be functions $f : E_1 \subset \mathbb{C} \longrightarrow \mathbb{C}$ differentiable at point $z \in E_1$ and $g : E_2 \subset \mathbb{C} \longrightarrow \mathbb{C}$ differentiable at point $w = f(z) \in E_2$, with respective differentials

$$df(z) : T_z\mathbb{C} \longrightarrow T_w\mathbb{C}$$
$$dg(w) : T_w\mathbb{C} \longrightarrow T_{g(w)}\mathbb{C}$$

Then, the composite function $g \circ f : E_1 \longrightarrow \mathbb{C}$ is differentiable at z, and $d(g \circ f)(z) : T_z\mathbb{C} \longrightarrow T_{g \circ f(z)}\mathbb{C}$ is

$$d(g \circ f)(z) = dg(w) \circ df(z) \iff (g \circ f)'(z) = g'\big(f(z)\big) \circ f'(z)$$

2.4.1 Power Series Representation of a Function

Definition 2.13 (Holomorphic Function)

If function $f : E \subset \mathbb{C} \longrightarrow \mathbb{C}$ is (complex) differentiable at a point $z_0 \in E$, then f is said to be **holomorphic at** z_0.

We recall the diagram that summarizes the conditions of differetiability and analyticity of a function f over the field $\mathbb{R}$.

f infinitely differentiable at x_0 $\iff$ Taylor series of f exists at x_0

Taylor series converges at x_0

Taylor series converges to f at x_0 $\iff$ f is analytic

In the theory of functions of a complex variable we actually have a remarkable theorem that does not have an analogue for functions over $\mathbb{R}$.

Theorem 2.5 (Analyticity of Differentiable Functions over $\mathbb{C}$)

If a function $f : E \subset \mathbb{C} \longrightarrow \mathbb{C}$ is differentiable in a neighborhood of a point $z_0 \in E$, then it is analytic at that point. In other words,

$$f \text{ is holomorphic at } z_0 \implies f \text{ is analytic at } z_0$$

This means that the conditions in the diagram above all are equivalent! Visually,

f is differentiable at z_0 $\iff$ f is holomorphic at z_0

$\Updownarrow$

f infinitely differentiable at z_0 $\iff$ Taylor series of f exists at z_0

$\Updownarrow$

Taylor series converges at z_0

$\Updownarrow$

Taylor series converges to f at z_0 $\iff$ f is analytic

This is certainly an amazing fact, since it then follows from the theorem that if a function $f(z)$ has one derivative $f'(z)$ in a neighborhood of a point, it also has derivatives of all orders in that neighborhood.

2.4.2 Algebraic Closedness of the Field $\mathbb{C}$

Definition 2.14 (Algebraically Closed Field)

A field $\mathbb{F}$ is **algebraically closed** if every nonconstant polynomial in $\mathbb{F}[x]$ (the polynomial ring with coefficients in $\mathbb{F}$) has a root in $\mathbb{F}$.

Theorem 2.6 (Fundamental Theorem of Algebra)

$\mathbb{C}$ is algebraically closed. That is, every polynomial

$$P(z) \equiv c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n$$

of degree $n \geq 1$ with complex coefficients $c_i \in \mathbb{C}$ $(i = 0, 1, \ldots, n)$ has a root in $\mathbb{C}$. This immediately implies that every polynomial $P(z)$ admits a representation (unique up to the order of the factors) in the form

$$P(z) = c_n (z - z_1)(z - z_2) \ldots (z - z_n)$$

where $z_1, \ldots, z_n \in \mathbb{C}$ not necessarily all distinct.

We can also prove the interesting property about zeroes of polynomials in $\mathbb{R}[x]$.

Corollary 2.4 (Complex Conjugate Roots of Real Polynomials)

Given a polynomial with real coefficients

$$P(z) \equiv a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

P, as we know, does not always have real roots (e.g. $P(x) = x^2 + 1$). However, we state that

$$\text{if } P(z_0) = 0, \text{ then } P(\overline{z}_0) = 0$$

Therefore, every polynomial P with real coefficients can be expanded as a product of linear and quadratic polynomial with real coefficients.

Proof.

We can see from the properties of complex numbers that

$$\begin{aligned}
\overline{(z_1 + z_2)} &= \overline{z_1} + \overline{z_2} \\
\overline{(z_1 \cdot z_2)} &= \overline{(r_1 e^{i\varphi_1} \cdot r_2 e^{i\varphi_2})} \\
&= \overline{r_1 r_2 e^{i(\varphi_1 + \varphi_2)}} = r_1 r_2 e^{-i(\varphi_1 + \varphi_2)} \\
&= r_1 e^{-i\varphi_1} \cdot r_2 e^{-i\varphi_2} = \overline{z}_1 \cdot \overline{z}_2
\end{aligned}$$

Thus, if $P(z_0) = 0$, then

$$0 = \overline{P(z_0)} = \overline{a_0 + \ldots + a_n z_0^n} = \overline{a}_0 + \ldots + \overline{a}_n \overline{z}_0^n = a_0 + \ldots + a_n \overline{z}_0^n = P(\overline{z}_0)$$

and thus $P(\overline{z}_0) = 0$.

2.5 Primitives

Definition 2.15 (Primitive)

A function $F(x)$ is a **primitive** of a function $f(x)$ on an interval if F is differentiable on the interval and satisfies the equation

$$F'(x) = f(x)$$

or equivalently, if their respective differentials satisfy

$$dF(x) = f(x)\, dx$$

Lemma 2.8 ()

If $F_1(x)$ and $F_2(x)$ are two primitives of $f(x)$ on the same interval, then the difference $(F_1 - F_2)(x)$ is constant on that interval.

Example 2.3 ()

Both

$$F_1(x) \equiv \arctan x \text{ and } F_2(x) \equiv \operatorname{arccot} \frac{1}{x}$$

are primitives of $f(x) = \frac{1}{1+x^2}$. Indeed, we can see by direct calculation that in the domain $\mathbb{R} \setminus 0$,

$$F_1(x) - F_2(x) = \arctan x - \operatorname{arccot} \frac{1}{x} = \begin{cases} 0, & x > 0 \\ -\pi, & x < 0 \end{cases}$$

which is supported by the lemma.

Notice how given a function $f(x)$, the operation of finding its differential, denoted with d, gives us a new function of h, called the differential

$$df(x)(h)$$

Similarly, the operation of finding a primitive of function $f(x)$, denoted with the symbol $\int$, gives us a new function.

Definition 2.16 (Indefinite Integration)

The operation of finding a primitive of a certain function $f(x)$ is called **indefinite integration**, and the mathematical notation

$$\int f(x)\, dx$$

is called the **indefinite integral of** $f(x)$ on a given interval (f called the **integrand** and $f(x)\, dx$ called the **differential form**).

1. It immediately follows from the lemma that if $F(x)$ is any particular primitive of $f(x)$ on the interval, then on that interval
$$\int f(x)\, dx = F(x) + C$$
2. If $F'(x) = f(x)$ (that is, F is a primitive of f on some interval), then we have
$$d \int f(x)\, dx = dF(x) = F'(x)\, dx$$
3. It also follows that
$$\int dF(x) = \int F'(x)\, dx = F(x) + C$$

Theorem 2.7 (Basic Methods of Indefinite Integration)

The definition of the indefinite integral has three basic properties that can be used to solve indefinite integrals.

1. Linearity of the indefinite integral.
$$\int \big(\alpha u(x) + \beta v(x)\big)\, dx = \alpha \int u(x)\, dx + \beta \int v(x)\, dx + C$$

2. Integration by parts.
$$\int (uv)'\,dx = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx + C$$
3. Change of Variable, or U-substitution. Given that $F'(x) = f(x)$ on an interval I_x and $\varphi : I_t \longrightarrow I_x$ is a C^1 mapping of interval I_t into I_x, then
$$\int (f \circ \varphi)(t)\varphi'(t)\,dt = (F \circ \varphi)(t) + C$$